RELATIVE HOMOLOGY AND HIGHER CLUSTER-TILTING THEORY

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Incomplete Preliminary Version

Contents

1. Introduction

The main aim of these notes is to present some homological aspects of higher cluster tilting theory in triangulated categories. We concentrate at four themes:

(i) Characterizations and basic properties of cluster tilting subcategories.
(ii) The Gorenstein condition.
(iii) The Calabi-Yau condition.
(iv) Global dimension of non-stable cluster tilting subcategories.

In the following we describe the main results contained in the notes, the starting point of which are the two basic papers of Keller and Reiten [21, 22].

1.1. **Cluster-tilting subcategories.** Let $\mathcal{T}$ be a triangulated category with split idempotents. We fix a full subcategory $\mathcal{X}$ of $\mathcal{T}$ and we assume always that $\mathcal{X}$ is closed under isomorphisms, (finite) direct sums and direct summands.

Definition 1.1. For an integer $n \geqslant 1$, we say that $\mathcal{X}$ is $(n+1)$-**cluster tilting** if:

(i) $\mathcal{X}$ is functorially finite in $\mathcal{T}$.
(ii) $\mathcal{X} = \big\{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[i]) = 0,\ \ 1 \leqslant i \leqslant n\big\}$.
(iii) $\mathcal{X} = \big\{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0,\ \ 1 \leqslant i \leqslant n\big\}$.

1.2. **Some useful tools.** There are several tools for the study of an $(n+1)$-cluster tilting subcategory $\mathcal{X}$ of $\mathcal{T}$. For the moment, let $\mathcal{X}$ be an arbitrary full subcategory of $\mathcal{T}$.

1. Associated with $\mathcal{X}$ are the full subcategories, $k \geqslant 1$:

$$\mathcal{X}_k^{\top} = \big\{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[i]) = 0,\ \ 1 \leqslant i \leqslant k\big\} \qquad \text{and} \qquad {}^{\top}_{k}\mathcal{X} = \big\{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0,\ \ 1 \leqslant i \leqslant k\big\}$$

We call $\mathcal{X}$ n-**rigid** if $\mathcal{X} \subseteq \mathcal{X}_n^{\top}$ or equivalently $\mathcal{X} \subseteq {}^{\top}_{n}\mathcal{X}$. Note that we have a decreasing filtration of $\mathcal{T}$:

$$\mathcal{T} \ \supseteq\ \mathcal{X}_1^{\top} \ \supseteq\ \mathcal{X}_2^{\top} \ \supseteq\ \cdots \ \supseteq\ \mathcal{X}_k^{\top} \ \supseteq\ \cdots \tag{1.1}$$

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2. For any $k \geqslant 0$, we consider all morphisms $f : A \longrightarrow B$ in $\mathcal{T}$ such that $\mathcal{T}(\mathcal{X}[k], f) = 0$. The set of all such morphisms forms a subgroup $\mathsf{Gh}_{\mathcal{X}[k]}(A, B)$ of $\mathcal{T}(A, B)$, called the subgroup of $\mathcal{X}[k]$**-ghost maps** and in this way we obtain an ideal $\mathsf{Gh}_{\mathcal{X}[k]}(\mathcal{T})$. Then the product ideal

$$\mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T}) = \mathsf{Gh}_{\mathcal{X}}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[1]}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[2]}(\mathcal{T}) \circ \cdots \circ \mathsf{Gh}_{\mathcal{X}[k-1]}(\mathcal{T})$$

is defined, and clearly $\mathsf{Gh}^{[k+1]}_{\mathcal{X}}(A, B)$ consists of all maps $f : A \longrightarrow B$ which can be written as a composition $f = f_0 \circ f_1 \circ \cdots \circ f_{k-1}$, where $f_0 : A \longrightarrow B_0$ is $\mathcal{X}$-ghost, $f_1 : B_0 \longrightarrow B_1$ is $\mathcal{X}[1]$-ghost, $\cdots$, $f_{n-1} : B_{n-1} \longrightarrow B$ is $\mathcal{X}[k-1]$-ghost.

Then there is an increasing filtration of the Hom-functor $\mathcal{T}(-,-)$ of $\mathcal{T}$:

$$\cdots \subseteq \mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T}) \subseteq \cdots \subseteq \mathsf{Gh}^{[2]}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathsf{Gh}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathcal{T}(-,-) \tag{1.2}$$

3. The structure of the ideal $\mathsf{Gh}^{[k]}_{\mathcal{X}}(\mathcal{T})$ of $\mathcal{X}$-ghost maps is related to the structure of the **category** $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$ **of extensions** of the subcategories $\mathcal{X}[k]$. Recall that if $\mathscr{A}_i$, $i = 1, 2$, are full subcategories of $\mathcal{T}$, then $\mathscr{A}_1 \star \mathscr{A}_2$ is the full subcategory of $\mathcal{T}$ consisting of all direct summands of objects $C \in \mathcal{T}$ for which there exists a triangle $A_1 \longrightarrow C \longrightarrow A_2 \longrightarrow A_1[1]$, where $A_i \in \mathscr{A}_i$. The full subcategory $\mathscr{A}_1 \star \mathscr{A}_2 \star \cdots \star A_k$ is defined inductively fro full subcategories A_i, $1 \leqslant i \leqslant k$.

Then we have an increasing filtration of $\mathcal{T}$:

$$\mathcal{X} \subseteq \mathcal{X} \star \mathcal{X}[1] \subseteq \cdots \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k] \subseteq \cdots \subseteq \mathcal{T} \tag{1.3}$$

4. Finally an indispensable tool for the study of an $(n+1)$-cluster tilting subcategory $\mathcal{X}$ of $\mathcal{T}$ is the homological functor

$$\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}, \quad \mathsf{H}(A) = \mathcal{T}(-, A)|_{\mathcal{X}}$$

defined for any contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$, where we denote by $\mathsf{mod}\text{-}\mathcal{X}$ the category of coherent (or finitely presented) contravariant functors over $\mathcal{X}$. Note that an easy consequence of the fact that $\mathcal{X}$ is contravariantly finite in $\mathcal{T}$, is that $\mathsf{mod}\text{-}\mathcal{X}$ is abelian.

If $\mathcal{X}$ is 1-rigid, then the functor H is surjective on objects.

1.3. **Relative Homology in $\mathcal{T}$.** If $\mathcal{X}$ is a full contravariantly finite subcategory of $\mathcal{T}$, then for any object $A \in \mathcal{T}$, there exists a triangle

$$\Omega^1_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$$

where the middle map is a right $\mathcal{X}$-approximation of A. Note that the object $\Omega^1_{\mathcal{X}}(A)$ is uniquely determined in the stable category $\mathcal{T}/\mathcal{X}$. Inductively we define the object $\Omega^k_{\mathcal{X}}(A)$, $\forall k \geqslant 1$. If $\mathcal{X}$ is covariantly finite, then dually the objects $\Sigma^k_{\mathcal{X}}(A)$ are defined, $\forall k \geqslant 1$. The minimum k, or ∞, such that $\Omega^k_{\mathcal{X}}$ lies in $\mathcal{X}$, is well-defined, it is denoted by $\mathsf{gl.dim}_{\mathcal{X}}\, \mathcal{T}$ and is called the $\mathcal{X}$**-global dimension** of $\mathcal{T}$.

Then we have the following characterizations of when a full subcategory $\mathcal{X}$ of $\mathcal{T}$ is $(n+1)$-cluster tilting, for some $n \geqslant 1$.

Theorem A. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$, and $n \geqslant 1$. Then the following are equivalent.*

(i) *$\mathcal{X}$ is a $(n+1)$-cluster tilting subcategory of $\mathcal{T}$.*
(ii) *$\mathcal{X}$ is contravariantly finite and $\mathcal{X} = \mathcal{X}^{\top}_n$.*
(iii) *$\mathcal{X}$ is covariantly finite and $\mathcal{X} = {}^{\top}_{n}\mathcal{X}$.*
(iv) *$\mathcal{X}$ is contravariantly finite and both $\mathcal{X}$ and $\mathcal{X}^{\top}_n$ are n-rigid.*
(v) *$\mathcal{X}$ covariantly finite and both $\mathcal{X}$ and ${}^{\top}_{n}\mathcal{X}$ are n-rigid.*
(vi) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{gl.dim}_{\mathcal{X}}\, \mathcal{T} = n$.*
(vii) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.*
(viii) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{Gh}^{[n+1]}(\mathcal{T}) = 0$.*
(ix) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and, $\forall A \in \mathcal{T}$: $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$.*
(x) *$\mathcal{X}$ is covariantly (or contravariantly) finite n-rigid and, $\forall A \in \mathcal{T}$: $\Sigma^n_{\mathcal{X}}(A) \in \mathcal{X}$.*
(xi) *$\mathcal{X}$ is contravariantly finite n-rigid, any object of $\mathcal{X}^{\top}_n[n+1]$ is injective in $\mathsf{mod}\text{-}\mathcal{X}$ and the functor $\mathsf{H} : \mathcal{X}^{\top}_n[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is full and reflects isomorphisms.*

If $\mathcal{X}$ is a $(n+1)$-cluster subcategory of $\mathcal{T}$, then $\mathcal{X}^{\top}_{n+1} = 0 = {}^{\top}_{n+1}\mathcal{X}$, the abelian category $\mathsf{mod}\text{-}\mathcal{X}$ has enough projectives and enough injectives, the functors $\mathcal{X}[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X} \longleftarrow \mathcal{X}$ are fully faithful and induce equivalences

$$\mathcal{X}[n+1] \xrightarrow{\approx} \mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X} \quad \textit{and} \quad \mathcal{X} \xrightarrow{\approx} \mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$$

1.4. **The Gorenstein condition.** Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. Recall the following invariants attached to $\mathscr{A}$:

$$\mathsf{silp}\mathscr{A} = \sup\{\mathsf{id}\, P \mid P \in \mathsf{Proj}\,\mathscr{A}\}, \qquad \mathsf{spli}\mathscr{A} = \sup\{\mathsf{pd}\, I \mid I \in \mathsf{Inj}\,\mathscr{A}\}$$

$$\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} := \max\big\{\mathsf{silp}\mathscr{A}, \mathsf{spli}\mathscr{A}\big\}$$

We call $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}$ the *Gorenstein dimension* of $\mathscr{A}$ and then $\mathscr{A}$ is called *Gorenstein* if $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} < \infty$. If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} \leqslant n < \infty$, then we say that $\mathscr{A}$ is *n-Gorenstein.*

Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geqslant 1$. If $n = 1$, then by a result of Keller-Reiten [21], the category $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein. However this fails for $n > 1$. To remedy this failure, we consider a strengthening of the notion of cluster tilting subcategories.

Definition 1.2. An $(n+1)$-cluster tilting subcategory $\mathcal{X}$ of $\mathcal{T}$ is called **t-strong**, where $1 \leqslant t \leqslant n$, if:

$$\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0, \qquad 1 \leqslant i \leqslant t$$

And $\mathcal{X}$ is called **strictly t-strong**, if $\mathcal{X}$ is t-strong but not $(t+1)$-strong.

Then we have the following result.

Theorem B. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$.*

(i) *If $n = 1$, then* $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 1$.
(ii) $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} = 0$ *if and only if* $\mathcal{X}$ *is n-strong.*
(iii) *Assume that $n \geqslant 2$ and $\mathcal{X}$ is $(n-k)$-strong, where $0 \leqslant k \leqslant n-1$. Then for $n \geqslant 2k-1$, the cluster tilted category* $\mathsf{mod}\text{-}\mathcal{X}$ *is Gorenstein:*

$$0 \leqslant k \leqslant \frac{n+1}{2} \qquad \Longrightarrow \qquad \mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$$

In particular:

(a) *If n is odd and $\mathcal{X}$ is $(\frac{n-1}{2})$-strong, then:* $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant \frac{n+1}{2}$.
(b) *If n is even and $\mathcal{X}$ is $(\frac{n+1}{2})$-strong, then:* $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant \frac{n-1}{2}$.

- *Moreover if $\mathcal{X}$ is strictly $(n-k)$-strong, then:* $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} = k$.

As a consequence we have the following which shows that if $\mathsf{mod}\text{-}\mathcal{X}$ has finite global dimension, then it can be realized as a full subcategory of $\mathcal{T}$, in some cases via a ∂-functor.

Theorem C. *Let $\mathcal{X}$ an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. If* $\mathsf{mod}\text{-}\mathcal{X}$ *has finite global dimension, and $n \geqslant 2k$, then* $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$ *and there is an equivalence*

$$\mathcal{T}(-,?)|_{\mathcal{X}} \;:\; \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\big) \cap \mathcal{X}_k^{\top}[k+1] \;\xrightarrow{\approx}\; \mathsf{mod}\text{-}\mathcal{X}$$

If $k = 1$, then the induced full embedding $\mathsf{T} : \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathcal{T}$ *is a ∂-functor, which extends uniquely to an additive functor* $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{X}) \longrightarrow \mathcal{T}$ *commuting with the shifts.*

1.5. **The Calabi-Yau condition.** Assume that the triangulated category $\mathcal{T}$ is k-linear with finite-dimensional Hom-spaces over a field k. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geqslant 1$. We assume that $\mathcal{T}$ is $(n+1)$-*Calabi-Yau.* This roughly means that there are natural isomorphisms:

$$\mathsf{D}\,\mathsf{Hom}_{\mathcal{T}}(A, B) \;\xrightarrow{\cong}\; \mathsf{Hom}_{\mathcal{T}}(B, A[n+1])$$

$\forall A, B \in \mathcal{T}$, where D denotes duality with respect to the base field k.

If $n = 1$, and $\mathcal{T}$ is 2-Calabi-Yau, then by a result of Keller-Reiten [22], the stable category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ of the Gorenstein-projective (or Cohen-Macaulay) objects of the 1-Gorenstein category $\mathsf{mod}\text{-}\mathcal{X}$ is 3-Calabi-Yau.

Assuming that $n \geqslant 1$.

Theorem D. *Assume that $\mathcal{T}$ is $(n+1)$-Calabi-Yau over a field k, and $\mathcal{X}$ is an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $0 \leqslant k \leqslant n-1$. Then* $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ *is $(n+2)$-Calabi-Yau in the following cases:*

(i) *$0 \leqslant k \leqslant 1$, or,*
(ii) *$2 \leqslant k \leqslant \frac{n+1}{2}$, and any object $\mathcal{T}(-,C)|_{\mathcal{X}}$ in* $\mathsf{mod}\text{-}\mathcal{X}$, *where $C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, has finite projective or injective dimension.*

1.6. **Global dimension of non stable cluster tilting subcategories.** Let $\mathscr{A}$ be an abelian category with enough projective objects. We consider full subcategories $\mathcal{M}$ of $\mathscr{A}$, closed under isomorphisms, finite direct sums and direct summands.

In analogy with the triangulated case, we define for any full subcategory $\mathcal{M} \subseteq \mathscr{A}$:

$$\mathcal{M}_n^{\perp} = \big\{A \in \mathscr{A} \ \mid \ \mathsf{Ext}^k(\mathcal{M}, A) = 0,\ 1 \leqslant k \leqslant n\big\} \quad \text{and} \quad {}^{\perp}_{n}\mathcal{M} = \big\{A \in \mathscr{A} \ \mid \ \mathsf{Ext}^k(A, \mathcal{M}) = 0,\ 1 \leqslant k \leqslant n\big\}$$

Then $\mathcal{M}$ is called n-**rigid** if $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$ or equivalently $\mathcal{M} \subseteq {}^{\perp}_{n}\mathcal{M}$.

We are interested in the case $\mathcal{M}$ is a lift of a cluster tilting subcategory $\mathcal{X}$ of the triangulated category $\underline{\mathsf{Gproj}}\,\mathscr{A}$ of Gorenstein-projective objects of $\mathscr{A}$, in the sense that $\mathcal{M} = \pi^{-1}(\mathcal{X})$, where $\pi : \mathsf{GProj}\,\mathscr{A} \longrightarrow \underline{\mathsf{GProj}}\,\mathscr{A}$ is the natural projection functor. Note that then $\mathcal{M}$ contains the projectives and at the level of coherent functors we have an inclusion $\mathsf{mod}\text{-}\mathcal{X} \subseteq \mathsf{mod}\text{-}\mathcal{M}$.

Let $\mathcal{U}$, $\mathcal{V}$ be full subcategories of $\mathscr{A}$. Then we define $\mathcal{U} \diamond \mathcal{V}$ to be the full subcategory

$$\mathcal{U} \diamond \mathcal{V} = \mathsf{add}\{A \in \mathscr{A} \ \mid \ \exists \text{ an exact sequence}: \ \ 0 \longrightarrow U \longrightarrow A \longrightarrow V \longrightarrow 0, \ \ \text{where } U \in \mathcal{U} \ \text{and} \ V \in \mathcal{V}\}$$

Inductively we define $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$, $\forall n \geqslant 1$, for full subcategories $\mathcal{U}_i$ of $\mathscr{A}$. Finally we denote by $\mathsf{Proj}^{\leqslant k}\,\mathscr{A}$, the full subcategory of $\mathscr{A}$ consisting of all objects with projective dimension $\leqslant k$.

Theorem E. *Let $\mathscr{A}$ be an abelian category with enough projectives. Let $\underline{\mathcal{X}}$ be a full subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$ and set $\mathcal{M} = \pi^{-1}\underline{\mathcal{X}}$. Then the following are equivalent.*

(i) *$\mathscr{A}$ is Gorenstein and $\underline{\mathcal{X}}$ is an $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$.*
(ii) *$\mathcal{M}$ is contravariantly finite in $\mathscr{A}$ and $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$ and $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} < \infty$.*

If (i) *holds and $\underline{\mathcal{X}} \neq 0$, then ${}^{\perp}_{n}\mathcal{M} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$, we have an equality*

$$\mathscr{A} = \mathcal{M} \diamond \Omega^{-1}\mathcal{M} \diamond \cdots \diamond \Omega^{-n}\mathcal{M} \diamond \mathsf{Proj}^{\leqslant d}\mathscr{A} \tag{1.4}$$

where $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} = d$, and $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M}$ is bounded as follows:

$$n+2 \ \ \leqslant \ \ \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \ \ \leqslant \ \ \max\big\{n, \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}\big\} + 3 \tag{1.5}$$

Moreover $\mathsf{pd}_{\mathsf{mod}\text{-}\mathcal{M}}\,F = n+2$, $\forall F \in \mathsf{mod}\text{-}\mathcal{X}$, $F \neq 0$, and:

(a) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} < n$, then: $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = n+2$.*
(b) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} = n$, then: $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \ \in \ \{n+2, n+3\}$.*
(c) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} > n$, then: $n+2 \ \leqslant \ \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \ \leqslant \ \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} + 3$.*

If $\underline{\mathsf{GProj}}\,\mathscr{A}$ is $(n+1)$-Calabi-Yau, then we have relative $(n+2)$-Calabi-Yau duality [21]*: for any object $F \in \mathsf{mod}\text{-}\mathcal{X} \subseteq \mathsf{mod}\text{-}\mathcal{M}$, there is a natural isomorphism:*

$$\mathsf{D}\,\mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(F, -\big) \ \xrightarrow{\ \cong\ } \ \mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(-, F[n+2]\big)$$

In particular, for any two objects $F, G \in \mathsf{mod}\text{-}\mathcal{M}$ with $F \in \mathsf{mod}\text{-}\mathcal{X} \subseteq \mathsf{mod}\text{-}\mathcal{M}$:

$$\mathsf{D}\mathsf{Ext}^i_{\mathsf{mod}\text{-}\mathcal{M}}\big(F, G\big) \ \xrightarrow{\ \cong\ } \ \mathsf{Ext}^{n-i+2}_{\mathsf{mod}\text{-}\mathcal{M}}\big(G, F\big), \qquad i \in \mathbb{Z}$$

As a consequence we have the following, see [21] for the case $n = 1$.

Theorem F. *Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. For a full subcategory $\mathcal{M} \subseteq \mathsf{GProj}\,\mathscr{A}$, the following are equivalent.*

(i) *$\mathscr{A}$ is Frobenius and $\underline{\mathcal{M}}$ is an $(n+1)$-cluster tilting subcategory of $\underline{\mathscr{A}}$.*
(ii) *$\mathcal{M}$ is contravariantly finite in $\mathscr{A}$, contains the projectives, and $\mathcal{M}_n^{\perp} = \mathcal{M}$.*
(iii) *$\mathcal{M}$ is covariantly finite in $\mathscr{A}$, contains the injectives, and ${}^{\perp}_{n}\mathcal{M} = \mathcal{M}$.*
(iv) *$\mathcal{M}$ is n-rigid, contravariantly in $\mathscr{A}$ and contains the projectives, or covariantly finite and contains the injectives, and $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = n+2$.*

In particular if $\mathscr{A}$ is a Krull-Schmidt Frobenius abelian category and $\mathcal{X}$ is an $(n+1)$-cluster subcategory of $\underline{\mathscr{A}}$ which is of finite representation type, then

$$\mathsf{rep.dim}\,\mathscr{A} \leqslant n+2$$

The invariant $\mathsf{rep.dim}\,\mathscr{A}$ above is Auslander's **representation dimension**, which in this context is defined to be the minimum of the global dimensions $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathscr{A}}(T)$, where T runs over all objects such that the subcategory $\mathsf{add}\,T$ contains the projectives and the injectives and moreover admits weak kernels and weak cokernels, equivalently the ring $\mathsf{End}_{\mathscr{A}}(T)$ is coherent.

2. Relative Homology

Throughout the paper we denote by $\mathcal{T}$ a triangulated category with split idempotents. We fix a full additive contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$ which is closed under isomorphisms and direct summands.

Let A be in $\mathcal{T}$ and consider triangles

$$\Omega^{t+1}_{\mathcal{X}}(A) \xrightarrow{g^t_A} X^t_A \xrightarrow{f^t_A} \Omega^t_{\mathcal{X}}(A) \xrightarrow{h^t_A} \Omega^{t+1}_{\mathcal{X}}(A)[1] \qquad (T^t_A)$$

where $\Omega^0_{\mathcal{X}}(A) := A$, and the middle map $X^t_A \longrightarrow \Omega^t_{\mathcal{X}}(A)$ to be a right $\mathcal{X}$-approximation of $\Omega^{t-1}_{\mathcal{X}}(A)$. Applying the homological functor $\mathsf{H}\colon \mathcal{T} \longrightarrow \mathsf{Mod}\text{-}\mathcal{X}$, defined by $\mathsf{H}(A) = \mathcal{T}(-,A)|_{\mathcal{X}}$, to the above triangles we have exact sequences $\mathsf{H}(\Omega^{t+1}_{\mathcal{X}}A) \longrightarrow \mathsf{H}(X^t_A) \longrightarrow \mathsf{H}(\Omega^t_{\mathcal{X}}A) \longrightarrow 0$, $\forall t \geqslant 0$. In particular we have an exact sequence which is a projective presentation of $\mathsf{H}(A)$:

$$\mathsf{H}(X^1_A) \longrightarrow \mathsf{H}(X^0_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$$

It follows that $\mathsf{H}(A)$ is coherent and therefore H induces a homological representation functor

$$\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}, \quad \mathsf{H}(A) = \mathcal{T}(-,A)|_{\mathcal{X}}$$

Since $\mathcal{T}$, as a triangulated category, has weak kernels, and since $\mathcal{X}$ is contravariantly finite in $\mathcal{T}$, it is easy to see that $\mathcal{X}$ has weak kernels and therefore the category $\mathsf{mod}\text{-}\mathcal{X}$ is abelian with enough projectives and it is well-known that H induces an equivalence $\mathcal{X} \approx \mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$. Clearly we have

$$\mathsf{Ker}\,\mathsf{H} = \mathcal{X}^{\top} := \{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A) = 0\}$$

Remark 2.1. Dually if $\mathcal{X}$ is covariantly finite, then the category $\mathcal{X}\text{-}\mathsf{mod}$ of coherent covariant functors is abelian and we have a contravariant cohomological functor

$$\mathsf{H}^{\mathsf{op}} : \mathcal{T}^{\mathsf{op}} \longrightarrow \mathcal{X}\text{-}\mathsf{mod}, \quad \mathsf{H}(A) = \mathcal{T}(A,-)|_{\mathcal{X}}$$

Clearly then $\mathsf{Ker}\,\mathsf{H}^{\mathsf{op}} = {}^{\top}\mathcal{X} := \{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}) = 0\}$.

2.1. **Ghost and Cellular Towers.** Fix an object A in $\mathcal{T}$ and consider as before the triangles

$$\Omega^{t+1}_{\mathcal{X}}(A) \xrightarrow{g^t_A} X^t_A \xrightarrow{f^t_A} \Omega^t_{\mathcal{X}}(A) \xrightarrow{h^t_A} \Omega^{t+1}_{\mathcal{X}}(A)[1] \qquad (T^t_A)$$

Then, as in [8, Section 5], we may construct inductively a tower of triangles in $\mathcal{T}$ containing all the essential information concerning the homological behavior of A with respect to $\mathcal{X}$, that we need in the sequel. We proceed as follows: First we form the weak-push-out, in the sense of [8], of the triangle

$$\Omega^1_{\mathcal{X}}(A) \xrightarrow{g^0_A} X^0_A \xrightarrow{f^0_A} A \xrightarrow{h^0_A} \Omega^1_{\mathcal{X}}(A)[1] \qquad (T^0_A)$$

along the map $h^1_A : \Omega^1_{\mathcal{X}}(A) \longrightarrow \Omega^2_{\mathcal{X}}(A)[1]$. Then we obtain a morphism of triangles

$$\begin{tikzcd}
\Omega^1_{\mathcal{X}}(A) \arrow[r, "g^0_A"] \arrow[d, "h^1_A"'] & X^0_A \arrow[r, "f^0_A"] \arrow[d, "\alpha^1_A"] & A \arrow[r, "h^0_A"] \arrow[d, equal] & \Omega^1_{\mathcal{X}}(A)[1] \arrow[d, "h^1_A[1]"] & (C^0_A) \\
\Omega^2_{\mathcal{X}}(A)[1] \arrow[r, "\beta^1_A"] & \mathsf{Cell}_1(A) \arrow[r, "\gamma^1_A"] & A \arrow[r, "\omega^1_A"] & \Omega^2_{\mathcal{X}}(A)[2] & (C^1_A)
\end{tikzcd}$$

Next consider the weak push-out of the lower triangle along the map $h^2[1] : \Omega^2_{\mathcal{X}}(A)[1] \longrightarrow \Omega^3_{\mathcal{X}}(A)[2]$:

$$\begin{tikzcd}
\Omega^2_{\mathcal{X}}(A)[1] \arrow[r, "\beta^1_A"] \arrow[d, "h^2_A[1]"'] & \mathsf{Cell}_1(A) \arrow[r, "\gamma^1_A"] \arrow[d, "\alpha^2_A"] & A \arrow[r, "\omega^1_A"] \arrow[d, equal] & \Omega^2_{\mathcal{X}}(A)[2] \arrow[d, "h^2_A[2]"] & (C^1_A) \\
\Omega^3_{\mathcal{X}}(A)[2] \arrow[r, "\beta^2_A"] & \mathsf{Cell}_2(A) \arrow[r, "\gamma^2_A"] & A \arrow[r, "\omega^2_A"] & \Omega^3_{\mathcal{X}}(A)[3] & (C^2_A)
\end{tikzcd}$$

Continuing in this way we obtain the following tower of triangles, henceforth denoted by $(C^{\bullet}_A)$:

$$
\begin{tikzcd}
\Omega^1_{\mathcal{X}}(A) \arrow[r, "\beta^0_A"] \arrow[d, "h^1_A"'] & \mathsf{Cell}_0(A) \arrow[r, "\gamma^0_A"] \arrow[d, "\alpha^1_A"] & A \arrow[r, "\omega^0_A"] \arrow[d, equal] & \Omega^1_{\mathcal{X}}(A)[1] \arrow[d, "h^1_A[1]"] & (C^0_A) \arrow[d] \\
\Omega^2_{\mathcal{X}}(A)[1] \arrow[r, "\beta^1_A"] \arrow[d, "h^2_A[1]"'] & \mathsf{Cell}_1(A) \arrow[r, "\gamma^1_A"] \arrow[d, "\alpha^2_A"] & A \arrow[r, "\omega^1_A"] \arrow[d, equal] & \Omega^2_{\mathcal{X}}(A)[2] \arrow[d, "h^2_A[2]"] & (C^1_A) \arrow[d] \\
\Omega^3_{\mathcal{X}}(A)[2] \arrow[r, "\beta^2_A"] \arrow[d] & \mathsf{Cell}_2(A) \arrow[r, "\gamma^2_A"] \arrow[d] & A \arrow[r, "\omega^2_A"] \arrow[d, equal] & \Omega^3_{\mathcal{X}}(A)[3] \arrow[d] & (C^2_A) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \vdots \arrow[d] \\
\Omega^{n-1}_{\mathcal{X}}(A)[n-2] \arrow[r, "\beta^{n-2}_A"] \arrow[d, "h^{n-1}_A[n-2]"'] & \mathsf{Cell}_{n-2}(A) \arrow[r, "\gamma^{n-2}_A"] \arrow[d, "\alpha^{n-1}_A"] & A \arrow[r, "\omega^{n-2}_A"] \arrow[d, equal] & \Omega^{n-1}_{\mathcal{X}}(A)[n-1] \arrow[d, "h^{n-1}_A[n-1]"] & (C^{n-2}_A) \arrow[d] \\
\Omega^n_{\mathcal{X}}(A)[n-1] \arrow[r, "\beta^{n-1}_A"] \arrow[d] & \mathsf{Cell}_{n-1}(A) \arrow[r, "\gamma^{n-1}_A"] \arrow[d] & A \arrow[r, "\omega^{n-1}_A"] \arrow[d, equal] & \Omega^n_{\mathcal{X}}(A)[n] \arrow[d] & (C^{n-1}_A) \arrow[d] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}
\tag{2.1}
$$

where for convenience we set: $\mathsf{Cell}_0(A) := X^0_A$, $\omega^0_A := h^0_A$, $\beta^0_A := f^0_A$, and $\gamma^0_A := g^0_A$. We call the map $\gamma^n_A : \mathsf{Cell}_n(A) \longrightarrow A$ the n^{th}-**cellular approximations** of A and the induced tower

$$
\begin{aligned}
A = \mathsf{Cell}_0(A) \xrightarrow{\alpha^1_A} \mathsf{Cell}_1(A) \xrightarrow{\alpha^2_A} \mathsf{Cell}_2(A) \longrightarrow \cdots \\
\cdots \longrightarrow \mathsf{Cell}_{n-1}(A) \xrightarrow{\alpha^{n-1}_A} \mathsf{Cell}_n(A) \longrightarrow \cdots
\end{aligned}
\tag{$\mathsf{Cell}^\bullet_A$}
$$

the **cellular tower** of A with respect to $\mathcal{X}$, and the tower of objects

$$
\begin{aligned}
A = \Omega^0_{\mathcal{X}}(A) \xrightarrow{h^0_A} \Omega^1_{\mathcal{X}}(A)[1] \xrightarrow{h^1_A[1]} \Omega^2_{\mathcal{X}}(A)[2] \longrightarrow \cdots \\
\cdots \longrightarrow \Omega^{n-1}_{\mathcal{X}}(A)[n-1] \xrightarrow{h^{n-1}_A[n-1]} \Omega^n_{\mathcal{X}}(A)[n] \longrightarrow \cdots
\end{aligned}
\tag{$\mathsf{Gh}^\bullet_A$}
$$

is called the **Ghost tower**, or an **Adams resolution**, of A with respect to $\mathcal{X}$.

Remark 2.2. (i) By the construction of the cellular tower $(C^\bullet_A)$ we have triangles, $\forall t \geqslant 1$:

$$
X^t_A[t-1] \longrightarrow \mathsf{Cell}_{t-1}(A) \xrightarrow{\alpha^t_A} \mathsf{Cell}_t(A) \longrightarrow X^t_A[t] \tag{2.2}
$$

$$
X^t_A[t] \longrightarrow \Omega^t_{\mathcal{X}}(A)[t]) \xrightarrow{h^t_A[t]} \Omega^{t+1}_{\mathcal{X}}(A)[t+1] \longrightarrow \Omega^t_{\mathcal{X}}(A)[t+1]) \tag{2.3}
$$

Since $\mathsf{Cell}_0(A) = X^0_A \in \mathcal{X}$, it follows that we have a triangle $X^0_A \longrightarrow \mathsf{Cell}_1(A) \longrightarrow X^1_A[1] \longrightarrow X^0_A[1]$ and therefore $\mathsf{Cell}_1(A) \in \mathcal{X} \star \mathcal{X}[1]$. By induction it follows that:

$$
\mathsf{Cell}_t(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t], \quad \forall t \geqslant 0
$$

(ii) Splicing the triangles (T^t_A) we obtain a complex

$$
\cdots \longrightarrow X^n_A \xrightarrow{\varepsilon^n_A} X^{n-1}_A \longrightarrow \cdots \longrightarrow X^1_A \xrightarrow{\varepsilon^1_A} X^0_A \xrightarrow{f^0_A} A \longrightarrow 0 \tag{$X^\bullet_A$}
$$

where $\varepsilon^n_A = f^n_A \circ g^{n-1}_A$, $\forall n \geqslant 1$. Note that the complex $(X^\bullet_A)$ does not necessarily becomes exact after the application of the functor H.

(iii) If $\Omega^n_{\mathcal{X}}(A)$ lies in $\mathcal{X}$, then $\Omega^{n-t}_{\mathcal{X}}(A)$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t]$, and we may choose $X^n_A = \Omega^n_{\mathcal{X}}(A)$. As a consequence we may choose $\Omega^{n+t}_{\mathcal{X}}(A) = 0$, $\forall t \geqslant 1$.

2.2. **Cellular and Co-Ghost Cotowers.** Dually if $\mathcal{X}$ is covariantly finite in $\mathcal{T}$, then for any object $A \in \mathcal{T}$ we may construct inductively triangles (T_t^A), $t \geqslant 0$:

$$\Sigma_{\mathcal{X}}^{t+1}(A)[-1] \xrightarrow{h_t^A} \Sigma_{\mathcal{X}}^{t}(A) \xrightarrow{f_t^A} X_t^A \xrightarrow{g_t^A} \Sigma_{\mathcal{X}}^{t+1}(A) \qquad (T_t^A)$$

where the middle map is a left $\mathcal{X}$-approximation of $\Sigma^t(A)$, and $\Sigma_{\mathcal{X}}^0(A) := A$. As before we may construct inductively a cotower of (morphisms of) triangles, henceforth denoted by $(C_\bullet^A)$, $\forall t \geqslant 0$:

$$\begin{array}{ccccccccc}
\Sigma_{\mathcal{X}}^{t}(A)[-t] & \xrightarrow{\omega_t^A} & A & \xrightarrow{\beta_t^A} & \mathsf{Cell}^{t-1}(A) & \xrightarrow{\gamma_t^A} & \Sigma_{\mathcal{X}}^{t}(A)[-t+1] & & (C_t^A) \\
{\scriptstyle h_{t-1}^A[-t+1]}\downarrow & & \| & & \downarrow{\scriptstyle \alpha_{t-1}^A} & & \downarrow{\scriptstyle h_{-t+2}^A} & & \downarrow \\
\Sigma_{\mathcal{X}}^{t-1}(A)[-t+1] & \xrightarrow{h_{t-1}^A} & A & \xrightarrow{\beta_{t-1}^A} & \mathsf{Cell}^{t-2}(A) & \xrightarrow{\gamma_{t-1}^A} & \Sigma_{\mathcal{X}}^{t-1}(A)[-t+2] & & (C_{t-1}^A)
\end{array}$$

where we set: $\mathsf{Cell}^0(A) := X_0^A$, $\omega_0^A := h_0^A$, $\beta_0^A := f_0^A$, and $\gamma_0^A := g_0^A$. The inverse tower of objects

$$\cdots \longrightarrow \mathsf{Cell}^{n+1}(A) \xrightarrow{\alpha_{n+1}^A} \mathsf{Cell}^{n}(A) \xrightarrow{\alpha_n^A} \mathsf{Cell}^{n-1}(A) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathsf{Cell}^{2}(A) \xrightarrow{\alpha_2^A} \mathsf{Cell}^{1}(A) \xrightarrow{\alpha_1^A} \mathsf{Cell}^{0}(A) = A \qquad (\mathsf{Cell}_\bullet^A)$$

is called the **cellular cotower** of A with respect to $\mathcal{X}$, and the cotower of objects

$$\cdots \longrightarrow \Sigma_{\mathcal{X}}^{n+1}(A)[-n-1] \xrightarrow{h_n^A[-n]} \Sigma_{\mathcal{X}}^{n}(A)[-n] \xrightarrow{h_{n-1}^A[-n+1]} \Sigma_{A}^{n-1}[-n+1] \longrightarrow \cdots$$
$$\cdots \longrightarrow \Sigma_{\mathcal{X}}^{2}(A)[-2] \xrightarrow{h_1^A[-1]} \Sigma_{\mathcal{X}}^{1}(A)[-1] \xrightarrow{h_0^A} \Sigma_{A}^{0} = A \qquad (\mathsf{Gh}_\bullet^A)$$

is called the **co-ghost cotower**, or an **Adams coresolution**, of A with respect to $\mathcal{X}$.

Remark 2.3. By the construction of the cellular tower $\mathsf{Cell}_\bullet^A$ we have triangles, $\forall n \geqslant 1$:

$$X_n^A[-n] \longrightarrow \mathsf{Cell}^n(A) \longrightarrow \mathsf{Cell}^{n-1}(A) \longrightarrow X_n^A[-n+1] \qquad (2.4)$$

Since $\mathsf{Cell}^0(A) = X_0^A \in \mathcal{X}$, it follows that we have a triangle $X_1^A[-1] \longrightarrow \mathsf{Cell}^1(A) \longrightarrow X_0^A \longrightarrow X_1^A$ and therefore $\mathsf{Cell}^1(A) \in \mathcal{X}[-1] \star \mathcal{X}$. By induction it follows that:

$$\mathsf{Cell}^t(A) \in \mathcal{X}[-t] \star \mathcal{X}[-t+1] \star \cdots \star \mathcal{X}[-1] \star \mathcal{X}, \qquad \forall n \geqslant 0$$

2.3. **The functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$.** We call an additive functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between additive categories $\mathscr{A}$ and $\mathscr{B}$ **almost full** if for any map $\alpha : F(A) \longrightarrow F(B)$ in $\mathscr{B}$, there are objects A^*, B^* in $\mathscr{A}$ and maps $\alpha^* : A^* \longrightarrow B^*$, $\omega_A : A^* \longrightarrow A$ and $\omega_B : B^* \longrightarrow B$, such that the maps $F(\omega_A) : F(A^*) \longrightarrow F(A)$ and $F(\omega_B) : F(B^*) \longrightarrow F(B)$ are invertible and the following square commutes:

$$\begin{array}{ccc}
F(A^*) & \xrightarrow{F(\alpha^*)} & F(B^*) \\
{\scriptstyle F(\omega_A)}\downarrow{\scriptstyle \cong} & & {\scriptstyle F(\omega_B)}\downarrow{\scriptstyle \cong} \\
F(A) & \xrightarrow{\alpha} & F(B)
\end{array}$$

Clearly F is full if and only if F is almost full and F is full on isomorphisms, i.e. any isomorphism $g : F(X) \longrightarrow F(Y)$ is of the form $F(f)$ for some map $f : X \longrightarrow Y$ in $\mathscr{A}$.

From now on the above notations will be used without further mentioning.

Now let as before $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$ and assume that $\mathcal{X}$ is closed under direct summands and isomorphisms.

Lemma 2.4. *Assume that* $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$.

(i) $\forall A \in \mathcal{T}$: $\mathsf{H}(\Omega_{\mathcal{X}}^k(A)[1]) = 0$, $\forall k \geqslant 1$. *Moreover the map* $\mathsf{H}(\gamma_A^1) : \mathsf{H}(\mathsf{Cell}_1(A)) \longrightarrow \mathsf{H}(A)$ *is invertible.*
(ii) *The functor* $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ *is almost full and essentially surjective.*

Proof. (i) Applying the homological functor H to the triangle $\Omega_{\mathcal{X}}^k(A) \longrightarrow X_A^{k-1} \longrightarrow \Omega_{\mathcal{X}}^{k-1}A \longrightarrow \Omega_{\mathcal{X}}^k(A)[1]$ and using that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we see that $\mathsf{H}(\Omega_{\mathcal{X}}^k(A)[1]) = 0$. On the other hand since $\mathsf{H}(h_A^0) = 0$, applying H to the triangle $(C_A^1) : \Omega_{\mathcal{X}}^2(A)[1] \longrightarrow \mathsf{Cell}_1(A) \xrightarrow{\gamma_A^1} A \xrightarrow{\omega_A^1} \Omega_{\mathcal{X}}^2(A)[2]$ and using that $\mathsf{H}(\Omega_{\mathcal{X}}^k(A)[1]) = 0$ and $\mathsf{H}(\omega_A^1) = \mathsf{H}(h_A^0) \circ \mathsf{H}(h_A^1[1]) = 0$, we infer that the map $\mathsf{H}(\gamma_A^1)$ is invertible.

(ii) Let $\mathsf{H}(X^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow F \longrightarrow 0$ be a projective presentation of $F \in \mathsf{mod}\text{-}\mathcal{X}$. If $X^1 \longrightarrow X^0 \longrightarrow A \longrightarrow X^1[1]$ is a triangle in $\mathcal{T}$, then applying H and using that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we have that $\mathsf{H}(A) = F$, so H is essentially surjective. Let $\alpha : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ be a map in $\mathsf{mod}\text{-}\mathcal{X}$ and let $\mathsf{H}(X_A^1) \longrightarrow \mathsf{H}(X_A^0) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ and $\mathsf{H}(X_B^1) \longrightarrow \mathsf{H}(X_B^0) \longrightarrow \mathsf{H}(B) \longrightarrow 0$ be projective presentations of $\mathsf{H}(A)$ and $\mathsf{H}(B)$. Since $\mathsf{H}|_{\mathcal{X}}$ is full and $\mathsf{H}(\mathcal{X}) = \mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$, we get a map of projective presentations in $\mathsf{mod}\text{-}\mathcal{X}$

$$\begin{tikzcd}
\mathsf{H}(X_A^1) \arrow[r, "\mathsf{H}(\varepsilon_A^1)"] \arrow[d, "\mathsf{H}(\gamma)"'] & \mathsf{H}(X_A^0) \arrow[r, "\mathsf{H}(f_A^0)"] \arrow[d, "\mathsf{H}(\beta)"'] & \mathsf{H}(A) \arrow[r] \arrow[d, "\alpha"'] & 0 \\
\mathsf{H}(X_B^1) \arrow[r, "\mathsf{H}(\varepsilon_B^1)"] & \mathsf{H}(X_B^0) \arrow[r, "\mathsf{H}(f_B^0)"] & \mathsf{H}(B) \arrow[r] & 0
\end{tikzcd}$$

Using that $\mathsf{H}|_{\mathcal{X}}$ is faithful, we get a morphism of triangles

$$\begin{tikzcd}
X_A^1 \arrow[r, "\varepsilon_A^1"] \arrow[d, "\gamma"'] & X_A^0 \arrow[r, "\alpha_A^1"] \arrow[d, "\beta"'] & \mathsf{Cell}_1(A) \arrow[r] \arrow[d, "\alpha^*"'] & X_A^1[1] \arrow[d] \\
X_B^1 \arrow[r, "\varepsilon_B^1"] & X_B^0 \arrow[r, "\alpha_B^1"] & \mathsf{Cell}_1(B) \arrow[r] & X_B^1[1]
\end{tikzcd}$$

By the construction of the towers $(C_A^\bullet)$ and $(C_B^\bullet)$ we have: $\alpha_A^1 \circ \gamma_A^1 = f_A^0$ and $\alpha_B^1 \circ \gamma_B^1 = f_B^0$, and by part (i) the maps $\mathsf{H}(\gamma_A^1)$ and $\mathsf{H}(\gamma_B^1)$ are invertible. Consider the diagram:

$$\begin{tikzcd}
\mathsf{H}(X_A^0) \arrow[r, "\mathsf{H}(\alpha_A^1)"] \arrow[d, "\mathsf{H}(\beta)"'] & \mathsf{H}(\mathsf{Cell}_1(A)) \arrow[r, "\mathsf{H}(\gamma_A^1)"] \arrow[d, "\mathsf{H}(\alpha^*)"'] & \mathsf{H}(A) \arrow[d, "\alpha"'] \\
\mathsf{H}(X_B^0) \arrow[r, "\mathsf{H}(\alpha_B^1)"] & \mathsf{H}(\mathsf{Cell}_1(B)) \arrow[r, "\mathsf{H}(\gamma_B^1)"] & \mathsf{H}(B)
\end{tikzcd}$$

where the left square is commutative. Then $\mathsf{H}(\alpha_A^1) \circ \mathsf{H}(\gamma_A^1) \circ \alpha = \mathsf{H}(f_A^0) \circ \alpha = \mathsf{H}(\beta) \circ \mathsf{H}(f_B^0) = \mathsf{H}(\beta) \circ \mathsf{H}(\alpha_B^1) \circ \mathsf{H}(\gamma_B^1) = \mathsf{H}(\alpha_A^1) \circ \mathsf{H}(\alpha^*) \circ \mathsf{H}(\gamma_B^1)$. By Remark 2.2 the cone of α_A^1 lies in $\mathcal{X}[1]$, so $\mathsf{H}(\alpha_A^1)$ is an epimorphism. Then $\mathsf{H}(\gamma_A) \circ \alpha = \mathsf{H}(\alpha^*) \circ \mathsf{H}(\gamma_B^1)$ and the right square is commutative. Hence H is almost full. ☐

3. Ghosts and Extensions

Let as before $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. Our aim here is to analyze the structure of maps in $\mathcal{T}$ which are invisible by the functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$, in the sense of the following definition.

Definition 3.1. A map $f : A \longrightarrow B$ in $\mathcal{T}$ is called **$\mathcal{X}$-ghost** if $\mathcal{T}(\mathcal{X}, f) = 0$; equivalently $\mathsf{H}(f) = 0$.

We let $\mathsf{Gh}_{\mathcal{X}}(A, B)$ be the set of all $\mathcal{X}$-ghost maps between A and B. Clearly $\mathsf{Gh}_{\mathcal{X}}(A, B)$ is a subgroup of $\mathcal{T}(A, B)$ and it is easy to see that in this way we obtain an ideal $\mathsf{Gh}_{\mathcal{X}}(\mathcal{T})$ of $\mathcal{T}$. We denote by $\mathsf{Gh}_{\mathcal{X}}^{[n]}(A, B)$ the subset of $\mathcal{T}(A, B)$ consisting of all maps $f : A \longrightarrow B$ which can be written as a composition $f = f_0 \circ f_1 \circ \cdots \circ f_{n-1}$, where $f_0 : A \longrightarrow B_0$ is $\mathcal{X}$-ghost, $f_1 : B_0 \longrightarrow B_1$ is $\mathcal{X}[1]$-ghost, $\cdots$, $f_{n-1} : B_{n-1} \longrightarrow B$ is $\mathcal{X}[n-1]$-ghost. Hence $\mathsf{Gh}_{\mathcal{X}}^{[n]}(\mathcal{T})$ is the product of ideals $\mathsf{Gh}_{\mathcal{X}[i]}(\mathcal{T})$, $0 \leqslant i \leqslant n-1$:

$$\mathsf{Gh}_{\mathcal{X}}^{[n]}(\mathcal{T}) = \mathsf{Gh}_{\mathcal{X}}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[1]}(\mathcal{T}) \circ \mathsf{Gh}_{\mathcal{X}[2]}(\mathcal{T}) \circ \cdots \circ \mathsf{Gh}_{\mathcal{X}[n-1]}(\mathcal{T})$$

Proposition 3.2 (The Ghost Lemma). (i) *For a map $f : A \longrightarrow B$ in $\mathcal{T}$, the following are equivalent:*

(a) $f \in \mathsf{Gh}_{\mathcal{X}}^{[n]}(A, B)$.

(b) *There exists a map $g : \Omega_{\mathcal{X}}^n(A)[n] \longrightarrow B$ such that:*

$$f = (-1)^{n+1} h_A^0 \circ h_A^1[1] \circ \cdots \circ h_A^{n-1}[n-1] \circ g = \omega_A^{n-1} \circ g$$

(ii) *Let A be an object in $\mathcal{T}$, and consider the following statements:*

(a) $\Omega_{\mathcal{X}}^n(A) \in \mathcal{X}$.

(b) $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.

(c) $\mathsf{Gh}_{\mathcal{X}}^{[n+1]}(A, -) = 0$.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). *In particular: $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ if and only if $\mathsf{Gh}_{\mathcal{X}}^{[n+1]}(\mathcal{T}) = 0$.*

Proof. (i) Clearly if f is as in (b), then f lies in $\mathsf{Gh}_{\mathcal{X}}^{[n]}(A, B)$, since by construction $h_A^i[i]$ is $\mathcal{X}[i]$-ghost, $0 \leqslant i \leqslant n-1$, cf. (2.4). To show the converse, let $n = 1$ and $f : A \longrightarrow B$ be $\mathcal{X}$-ghost. Then the composition $X_A^0 \longrightarrow A \longrightarrow B$ is zero and therefore it factorizes through $h_A^0 : A \longrightarrow \Omega_{\mathcal{X}}^1(A)[1]$. If $n = 2$ and $f : A \longrightarrow B$ lies in $\mathsf{Gh}_{\mathcal{X}}^{[2]}(A, B)$, then f admits a factorization $f = f_0 \circ f_1$, where $f_0 : A \longrightarrow B_0$ is $\mathcal{X}$-ghost and $f_1 : B_0 \longrightarrow B$ is $\mathcal{X}[1]$-ghost. Then there exists a map $g_0 : \Omega_{\mathcal{X}}^1(A)[1] \longrightarrow B_0$ such that $f_0 = h_A^0 \circ g_0$. Since f_1 is $\mathcal{X}[1]$-ghost, so is $g_0 \circ f_1$ and therefore its composition with the map $X^1_A[1] \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$ is zero. Hence there exists a map $g : \Omega^2_{\mathcal{X}}(A)[2] \longrightarrow B$ such that $(-h^1_A[1]) \circ g = g_0 \circ f_1$. Then $f = f_0 \circ f_1 = h^0_A \circ g_0 \circ f_1 = h^0_A \circ (-h^1_A[1]) \circ g$. Hence $f = -(h^0_A \circ h^1_A[1]) \circ g = \omega^1_A \circ g$. The assertion for $n \geqslant 3$ follows by induction.

(ii) (a) $\Rightarrow$ (b) Since $\mathsf{Cell}_{n-1}(A)$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$, it follows that if $\Omega^n_{\mathcal{X}}(A)$ lies in $\mathcal{X}$, then A lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ as follows from the triangle (C^{n-1}_A) in the tower of triangles $(C^{\bullet}_A)$.

(b) $\Leftrightarrow$ (c) If $n = 0$, i.e. $A \in \mathcal{X}$, then clearly $\mathsf{Gh}^{[1]}_{\mathcal{X}}(A, -) = \mathsf{Gh}_{\mathcal{X}}(A, -) = 0$. Let $n = 1$, so that $A \in \mathcal{X} \star \mathcal{X}[1]$, i.e. there is a triangle $X_0 \xrightarrow{\alpha} A \xrightarrow{\beta} X_1[1] \longrightarrow X_0[1]$, where the X_i lie in $\mathcal{X}$. Let f be in $\mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B)$, i.e. f admits a factorization $f = f_0 \circ f_1$, where $f_0 : A \longrightarrow B_0$ is $\mathcal{X}$-ghost and $f_1 : B_0 \longrightarrow B$ be $\mathcal{X}[1]$-ghost. Then $f_0 = \beta \circ g$ for some $g : X_1[1] \longrightarrow B_0$. Since the composition $g \circ f_1 : X_1[1] \longrightarrow B$ is $\mathcal{X}[1]$-ghost, we have $g \circ f_1 = 0$. Then $f = f_0 \circ f_1 = \beta \circ g \circ f_1 = 0$. Hence $\mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B) = 0$. For $n \geqslant 3$ the assertion follows by induction. Conversely if $\mathsf{Gh}^{[n+1]}_{\mathcal{X}}(A, -) = 0$, then the map $\omega^n_A = h^0_A \circ h^1_A[1] \circ \cdots \circ h^{n-1}_A[n-1] \circ h^n_A[n]$ lies in $\mathsf{Gh}^{[n+1]}_{\mathcal{X}}\big(A, \Omega^{n+1}_{\mathcal{X}}(A)[n]\big)$, and therefore $\omega^n_A = 0$. This implies that A lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ as a direct summand of $\mathsf{Cell}_n(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. $\square$

Lemma 3.3. *For any $t \geqslant 0$, the map $\gamma^t_A : \mathsf{Cell}_t(A) \longrightarrow A$ is a right $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$-approximation of A. In particular $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$ is contravariantly finite in $\mathcal{T}$.*

Proof. Consider the triangle $(C^{t+1}_A) : \Omega^{t+1}_{\mathcal{X}}(A)[t] \xrightarrow{\beta^t_A} \mathsf{Cell}_t(A) \xrightarrow{\gamma^t_A} A \xrightarrow{\omega^t_A} \Omega^{t+1}_{\mathcal{X}}(A)[t+1]$. Let $g : C \longrightarrow A$ be a map, where C lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[t]$. Since the map ω^t_A lies in $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, \Omega^{t+1}_{\mathcal{X}}(A)[t+1])$, it follows that $g \circ \omega^t_A$ lies in $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(C, \Omega^{t+1}_{\mathcal{X}}(A)[t+1])$. Since $C \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$, by the Ghost Lemma, we have $g \circ \omega^t_A = 0$ and therefore g factorizes through $\mathsf{Cell}_t(A)$, i.e. γ^t_A is a right $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t])$-approximation of A and $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$ is contravariantly finite in $\mathcal{T}$. $\square$

Lemma 3.4. *For any objects $A, B \in \mathcal{T}$ and any $t \geqslant 1$, we have an equality:*

$$\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, B) = \mathsf{Gh}_{\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]}(A, B)$$

Proof. If $f : A \longrightarrow B$ lies in $\mathsf{Gh}_{\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]}(A, B)$, then the composition $\gamma^t_A \circ f$ is zero since $\mathsf{Cell}_t(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$. Hence f factorizes through $\omega^t_A : A \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t+1]$, say as $f = \omega^t_A \circ g$. Since ω^t_A lies in $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, \Omega^{t+1}_{\mathcal{X}}(A)[t+1])$, f lies in $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, B)$. Hence $\mathsf{Gh}_{\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]}(A, B) \subseteq \mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, B)$. Conversely if f lies in $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, B)$, then by the Ghost Lemma we have $f = \omega^t_A \circ g$ for some $g : \Omega^{t+1}_{\mathcal{X}}(A)[t+1] \longrightarrow B$. This implies that $\gamma^t_A \circ f = 0$. Since, by Lemma 3.3, γ^t_A is a right $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$-approximation of A, we have $\mathcal{T}(C, f) = 0$, $\forall C \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$, so f lies in $\mathsf{Gh}_{\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]}(A, B)$. The last assertion is clear. $\square$

Remark 3.5. For any integer $n \geqslant 1$, we define $\mathsf{Gh}^{[-n]}_{\mathcal{X}}(A, B)$ to be the subgroup of $\mathcal{T}(A, B)$ consisting of all maps $f : A \longrightarrow B$ which can be written as a composition $f = f_{-n+1} \circ f_{-n+2} \circ \cdots \circ f_{-1} \circ f_0$, where $f_{-i} : B_{-i-1} \longrightarrow B_{-i}$ is $\mathcal{X}[-i]$-ghost, $A = B_{-n}$ and $B_0 = B$. Clearly then the map

$$\mathsf{Gh}^{[-n]}_{\mathcal{X}}(A, B) \longrightarrow \mathsf{Gh}^{[n]}_{\mathcal{X}}(A[n-1], B[n-1]), \qquad f \longrightarrow f[n-1]$$

is an isomorphism. It follows from Lemma 3.4 that we have an isomorphism $\mathsf{Gh}_{\mathcal{X}[-n] \star \cdots \star \mathcal{X}[-1] \star \mathcal{X}}(A, B) \cong \mathsf{Gh}_{\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]}(A[n], B[n])$.

Clearly if $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathsf{Gh}_{\mathcal{Y}}(A, B) \subseteq \mathsf{Gh}_{\mathcal{X}}(A, B)$. Hence the increasing filtration by subcategories of $\mathcal{T}$

$$\mathcal{X} \subseteq \mathcal{X} \star \mathcal{X}[1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2] \subseteq \cdots \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t] \subseteq \cdots \subseteq \mathcal{T} \tag{3.1}$$

induces a decreasing filtration of its Hom-functor of $\mathcal{T}$ by ghost ideals:

$$\cdots \subseteq \mathsf{Gh}^{[t+1]}_{\mathcal{X}}(\mathcal{T}) \subseteq \cdots \subseteq \mathsf{Gh}^{[2]}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathsf{Gh}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathcal{T}(-,-) \tag{3.2}$$

By the Ghost Lemma it follows that the above filtrations have the same length.

Proposition 3.6. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. Then there exists an exact sequence*

$$0 \longrightarrow \mathsf{Gh}_{\mathcal{X}}(A, B) \longrightarrow \mathcal{T}(A, B) \longrightarrow \mathsf{Hom}[\mathsf{H}(A), \mathsf{H}(B)] \longrightarrow \mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), B) \longrightarrow \mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[1]) \longrightarrow 0$$

for any objects $A, B \in \mathcal{T}$, where $\mathsf{H}_{A,B} : \mathcal{T}(A, B) \longrightarrow \mathsf{Hom}(\mathsf{H}(A), \mathsf{H}(B))$ is the canonical map.

Proof. We construct maps

$$\vartheta_{A,B} : \mathsf{Hom}(\mathsf{H}(A), \mathsf{H}(B)) \longrightarrow \mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), B) \quad \text{and} \quad \zeta_{A,B} : \mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), B) \longrightarrow \mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[1])$$

as follows. Let $\alpha : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ be a map and consider the triangle $\Omega^1_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. Then we have an exact sequence $\mathsf{H}(\Omega^1_{\mathcal{X}}(A)) \longrightarrow \mathsf{H}(X^0_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathcal{X}$. Then the composition $\mathsf{H}(f^0_A) \circ \alpha : \mathsf{H}(X^0_A) \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ is of the form $\mathsf{H}(\alpha^*)$ for a unique map $\alpha^* : X^0_A \longrightarrow B$. Define $\vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$. Clearly $g^0_A \circ \alpha^*$ is $\mathcal{X}$-ghost since $\mathsf{H}(g^0_A \circ \alpha^*) = \mathsf{H}(g^0_A) \circ \mathsf{H}(\alpha^*) = \mathsf{H}(g^0_A) \circ \mathsf{H}(f^0_A) \circ \alpha = 0$. Now in $\alpha \in \mathsf{Hom}(\mathsf{H}(A), \mathsf{H}(B))$ lies in $\mathsf{Ker}\,\vartheta_{A,B}$, then $\vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* = 0$, then $\alpha^* = f^0_A \circ \beta$ for some map $\beta : A \longrightarrow B$ and then $\mathsf{H}(f^0_A) \circ \alpha = \mathsf{H}(\alpha^*) = \mathsf{H}(f^0_A) \circ \mathsf{H}(\beta)$ and therefore $\alpha = \mathsf{H}(\beta)$. Conversely if $\alpha = \mathsf{H}(\beta)$, for some map $\beta : A \longrightarrow B$, then $\alpha^* = f^0_A \circ \beta$ and then $\vartheta_{A,B}(\alpha) = g^0_A \circ \alpha^* = g^0_A \circ f^0_A \circ \beta = 0$. Hence $\mathsf{Ker}\,\vartheta_{A,B} = \mathsf{Im}\,\mathsf{H}_{A,B}$. On the other hand, if $\alpha : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ is an $\mathcal{X}$-ghost map, then since $h^0_A[-1]$ is clearly $\mathcal{X}[-1]$-ghost and α is $\mathcal{X}$-ghost, it follows by the above Lemma that $h^0_A[-1] \circ \alpha \in \mathsf{Gh}_{\mathcal{X}[-1]\star\mathcal{X}}(A[-1], B)$, hence by Remark 3.5, we have a map $\zeta_{A,B} : \mathsf{Gh}_{\mathcal{X}}(\Omega_{\mathcal{X}}(A), B) \longrightarrow \mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[1])$, $\alpha \longmapsto \zeta_{A,B}(\alpha) = h^0_A \circ \alpha[1]$. We show that $\zeta_{A,B}$ is surjective. Let $\alpha : A \longrightarrow B[1]$ be a map in $\mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[1])$, so $\alpha = \beta \circ \gamma$, where $\beta \in \mathsf{Gh}_{\mathcal{X}}(A, C)$ and $\gamma \in \mathsf{Gh}_{\mathcal{X}[1]}(C, B[1])$. By the Ghost Lemma, $\beta = h^0_A \circ \rho$, for some map $\rho : \Omega^1_{\mathcal{X}}(A)[1] \longrightarrow C$. Then the map $(\rho \circ \gamma)[-1] : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ is $\mathcal{X}$-ghost and $\zeta_{A,B}((\rho \circ \gamma)[-1]) = \alpha$, so $\zeta_{A,B}$ is surjective. Finally we show that $\mathsf{Ker}\,\zeta_{A,B} = \mathsf{Im}\,\vartheta_{A,B}$. Let $\beta : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ be $\mathcal{X}$-ghost such that $\zeta_{A,B}(\beta) = h^0_A \circ \beta[1] = 0$. Then $\beta = g^0_A \circ \gamma$, for some map $\gamma : X^0_A \longrightarrow B$. Since $0 = \mathsf{H}(\beta) = \mathsf{H}(g^0_A) \circ \mathsf{H}(\gamma)$, there exists a unique map $\alpha : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ such that $\mathsf{H}(f^0_A) \circ \alpha = \mathsf{H}(\gamma)$. It follows that $\vartheta_{A,B}(\alpha) = g^0_A \circ \gamma = \beta$, i.e. $\mathsf{Ker}\,\zeta_{A,B} \subseteq \mathsf{Im}\,\vartheta_{A,B}$. Finally if $\delta : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ is in the image of $\vartheta_{A,B}$, then $\delta = g^0_A \circ \delta^*$ for some map $\delta^* : X^0_A \longrightarrow B$. Plainly $\zeta_{A,B}(\delta) = h^0_A \circ g^0_A[1] \circ \delta^*[1] = 0$, i.e. $\delta \in \mathsf{Ker}\,\zeta_{A,B}$. Hence $\mathsf{Ker}\,\zeta_{A,B} = \mathsf{Im}\,\vartheta_{A,B}$ and the sequence is exact. □

We say that a map $\alpha : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ is *realizable* (with respect to H), if $\alpha = \mathsf{H}(\alpha^*)$ for some map $\alpha^* : A \longrightarrow B$. The **obstruction group** $\mathcal{O}_{A,B}$ of the objects $A, B \in \mathcal{T}$ is defined as the cokernel of the natural map $\mathsf{H}_{A,B} : \mathcal{T}(A, B) \longrightarrow \mathsf{Hom}(\mathsf{H}(A), \mathsf{H}(B))$. Clearly $\mathcal{O}_{A,B} = 0$ if and only if any map $\mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ is realizable, and $\mathcal{O}_{A,B} = 0$, $\forall A, B \in \mathcal{T}$, if and only if H is full.

Corollary 3.7. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. If $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, then $\forall A \in \mathcal{X} \star \mathcal{X}[1]$ we have $\mathcal{O}_{A,B} = 0$, $\forall B \in \mathcal{T}$, i.e. there is an exact sequence:*

$$0 \longrightarrow \mathsf{Gh}_{\mathcal{X}}(A, B) \longrightarrow \mathcal{T}(A, B) \longrightarrow \mathsf{Hom}[\mathsf{H}(A), \mathsf{H}(B)] \longrightarrow 0$$

Moreover the functor H induces an equivalence: $\big(\mathcal{X} \star \mathcal{X}[1]\big)/\mathcal{X}[1] \xrightarrow{\approx} \mathsf{mod}\text{-}\mathcal{X}$.

Proof. If $A \in \mathcal{X} \star \mathcal{X}[1]$, then there is a triangle $X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow X_1[1]$, where $X_i \in \mathcal{X}$. Since $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, the map $X_0 \longrightarrow A$ is a right $\mathcal{X}$-approximation, hence $X_1 = \Omega^1_{\mathcal{X}}(A)$. It follows that $\mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), -) = 0$ and then by Proposition 3.6, the map $\mathsf{H}_{A,B} : \mathcal{T}(A, B) \longrightarrow \mathsf{Hom}[\mathsf{H}(A), \mathsf{H}(B)]$ is surjective, $\forall B \in \mathcal{T}$. In particular $\mathsf{H} : \mathcal{X} \star \mathcal{X}[1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is full. If $\alpha : A \longrightarrow B$ is $\mathcal{X}$-ghost, then by the Ghost Lemma, α factorizes through $h^1_A : A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. Since $\Omega^1_{\mathcal{X}}(A)[1] = X_1[1]$, we infer that $\mathsf{Gh}_{\mathcal{X}}(A, B)$ is the subgroup of all maps factorizing through an object from $\mathcal{X}[1]$. Finally let $F = \mathsf{H}(A)$ be in $\mathsf{mod}\text{-}\mathcal{X}$ and consider the triangle $(C^1_A) : \Omega^2_{\mathcal{X}}(A)[1] \xrightarrow{\beta^1_A} \mathsf{Cell}_1(A) \xrightarrow{\gamma^1_A} A \xrightarrow{\omega^1_A} \Omega^2_{\mathcal{X}}(A)[2]$. Since $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we have $\mathsf{H}(\Omega^2_{\mathcal{X}}(A)[1]) = 0$ and since ω^1_A is $\mathcal{X}$-ghost, we have $\mathsf{H}(\omega^1_A) = 0$. Hence the map $\mathsf{H}(\gamma^1_A) : \mathsf{H}(\mathsf{Cell}_1(A)) \longrightarrow \mathsf{H}(A)$ is invertible, i.e. $\mathsf{H} : \mathcal{X} \star \mathcal{X}[1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is surjective on objects. □

Proposition 3.8. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. Then for any object $A \in \mathcal{T}$ such that $\mathcal{T}(\mathcal{X}, A[-1]) = 0$, there exists a 7-term exact sequence:*

$$\begin{aligned} 0 \longrightarrow \mathsf{Gh}_{\mathcal{X}}(A, B) \longrightarrow \mathcal{T}(A, B) \xrightarrow{\mathsf{H}_{A,B}} \mathsf{Hom}\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\vartheta_{A,B}} \mathsf{Gh}_{\mathcal{X}}(\Omega_{\mathcal{X}}(A), B) \xrightarrow{\eta_{A,B}} \\ \mathsf{Gh}_{\mathcal{X}}(A, B[1]) \xrightarrow{\xi_{A,B}} \mathsf{Ext}^1\big(\mathsf{H}(A), \mathsf{H}(B)\big) \longrightarrow \mathcal{O}_{\Omega_{\mathcal{X}}(A),B} \longrightarrow 0 \end{aligned} \tag{3.3}$$

where $\mathsf{Im}(\vartheta_{A,B}) = \mathcal{O}_{A,B}$ and $\mathsf{Im}(\eta_{A,B}) \cong \mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[1])$.

Proof. Applying H to the triangle $\Omega_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega_{\mathcal{X}}(A)[1]$ and using that $\mathsf{H}(A[-1]) = 0$, we have an exact sequence $0 \longrightarrow \mathsf{H}(\Omega_{\mathcal{X}}(A)) \longrightarrow \mathsf{H}(X^0_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ and an exact commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{T}(A,B) \arrow[r] \arrow[d, "\mathsf{H}_{A,B}"'] & \mathcal{T}(X^0_A,B) \arrow[r, "\mathcal{T}(g^0_A,B)"] \arrow[d, "\mathsf{H}_{X^0_A,B}"', "\cong"] & \mathcal{T}(\Omega_{\mathcal{X}}(A),B) \arrow[r] \arrow[d, "\mathsf{H}_{\Omega_{\mathcal{X}}(A),B}"'] & \mathsf{Im}\,\mathcal{T}(h^0_A[-1],B) \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & (\mathsf{H}(A),\mathsf{H}(B)) \arrow[r] & (\mathsf{H}(X^0_A),\mathsf{H}(B)) \arrow[r] & (\mathsf{H}(\Omega_{\mathcal{X}}(A)),\mathsf{H}(B)) \arrow[r] & \mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B)) \arrow[r] & 0
\end{tikzcd}$$

and we know that $\mathsf{Ker}\,\mathsf{H}_{A,B} = \mathsf{Gh}_{\mathcal{X}}(A,B)$, $\mathsf{Coker}\,\mathsf{H}_{A,B} = \mathcal{O}_{A,B}$, and $\mathsf{Ker}\,\mathsf{H}_{\Omega_{\mathcal{X}}(A),B} = \mathsf{Gh}_{\mathcal{X}}(\Omega_{\mathcal{X}}(A),B)$, and $\mathsf{Coker}\,\mathsf{H}_{\Omega_{\mathcal{X}}(A),B} = \mathcal{O}_{\Omega_{\mathcal{X}}(A),B}$. Moreover the above diagram induces an exact sequence:

$$0 \longrightarrow \mathcal{O}_{A,B} \longrightarrow \mathsf{Gh}_{\mathcal{X}}(\Omega_{\mathcal{X}}(A),B) \longrightarrow \mathsf{Im}\,\mathcal{T}(h^0_A[-1],B) \xrightarrow{\phi} \mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B)) \longrightarrow \mathsf{Coker}\,\phi \longrightarrow 0$$

By chasing the above diagram we see easily that $\mathsf{Coker}\,\phi \cong \mathsf{Coker}\,\mathsf{H}_{\Omega_{\mathcal{X}}(A),B} = \mathcal{O}_{\Omega_{\mathcal{X}}(A),B}$ and $\mathsf{Im}\,\mathcal{T}(h^0_A[-1],B) = \mathsf{Gh}_{\mathcal{X}[-1]}(A[-1],B)$ and $\mathsf{Ker}\,\phi = \mathsf{Gh}_{\mathcal{X}[-1]\star\mathcal{X}}(A[-1],B)$. Since, by Remark 3.5, we have isomorphisms: $\mathsf{Gh}_{\mathcal{X}[-1]}(A[-1],B) \cong \mathsf{Gh}_{\mathcal{X}}(A,B[1])$ and $\mathsf{Gh}_{\mathcal{X}[-1]\star\mathcal{X}}(A[-1],B) \cong \mathsf{Gh}^{[-2]}_{\mathcal{X}}(A[-1],B) \cong \mathsf{Gh}^{[2]}_{\mathcal{X}}(A,B[1])$, (3.3) follows by splicing the above exact sequence with the exact sequence of Proposition 3.6. $\square$

Corollary 3.9. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$, and assume that $\mathcal{T}(\mathcal{X},A[-1]) = 0$.*

(i) *If $\Omega^2_{\mathcal{X}}(A) \in \mathcal{X}$, then, $\forall B \in \mathcal{T}$, there is an isomorphism*

$$\mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B)) \xrightarrow{\cong} \frac{\mathsf{Gh}_{\mathcal{X}}(A,B[1])}{\mathsf{Gh}^2_{\mathcal{X}}(A,B[1])}$$

(ii) *If $A \in \mathcal{X}\star\mathcal{X}[1]$ and $\mathcal{T}(\mathcal{X},\mathcal{X}[1]) = 0$, then, $\forall B \in \mathcal{T}$, there is an isomorphism:*

$$\mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B)) \xrightarrow{\cong} \mathsf{Gh}_{\mathcal{X}}(A,B[1])$$

(iii) *If $A,B \in \mathcal{X}\star\mathcal{X}[1]$ and $\mathcal{T}(\mathcal{X},\mathcal{X}[1]) = 0 = \mathcal{T}(\mathcal{X},\mathcal{X}[2])$, then there is an isomorphism:*

$$\mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B)) \xrightarrow{\cong} \mathcal{T}(A,B[1])$$

Proof. (i) Since $\Omega^2_{\mathcal{X}}(A) \in \mathcal{X}$, it follows that $\Omega^1_{\mathcal{X}}(A) \in \mathcal{X}\star\mathcal{X}[1]$. Then by Corollary 3.7 we have $\mathcal{O}_{\Omega^1_{\mathcal{X}}(A),B} = 0$, $\forall B \in \mathcal{T}$, and the assertion follows from (3.3).

(ii), (iii) If $A \in \mathcal{X}\star\mathcal{X}[1]$ and $\mathcal{T}(\mathcal{X},\mathcal{X}[1]) = 0$, then $\Omega_{\mathcal{X}}(A) \in \mathcal{X}$. Hence $\mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A),B) = 0 = \mathcal{O}_{\Omega^1_{\mathcal{X}}(A),B}$ and the map $\mathsf{Gh}_{\mathcal{X}}(A,B[1]) \longrightarrow \mathsf{Ext}^1(\mathsf{H}(A),\mathsf{H}(B))$ is invertible. Finally assume that in addition $\mathcal{T}(\mathcal{X},\mathcal{X}[2]) = 0$ and $B \in \mathcal{X}\star\mathcal{X}[1]$. Then there exists a triangle $X^1 \longrightarrow X^0 \longrightarrow B \longrightarrow X^1[1]$ where the X^j lie in $\mathcal{X}$. Applying H and using that $\mathcal{T}(\mathcal{X},\mathcal{X}[i]) = 0$, $1 \leqslant i \leqslant 2$, it follows that $\mathsf{H}(B[1]) = 0$. In particular any map $A \longrightarrow B[1]$ is $\mathcal{X}$-ghost. Hence by (ii), $\mathsf{Gh}_{\mathcal{X}}(A,B[1]) \cong \mathcal{T}(A,B[1])$ and (iii) follows from (ii). $\square$

Corollary 3.10. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. Let $t \geqslant 2$ and assume that $\mathcal{T}(\mathcal{X},\mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant t-1$. Then for any $A \in \mathcal{T}$ such that $\mathcal{T}(\mathcal{X},A[-i]) = 0$, $1 \leqslant i \leqslant t$, and any object $B \in \mathcal{T}$, there exists an exact sequence, $1 \leqslant i \leqslant t-1$:*

$$\begin{aligned} 0 \longrightarrow \mathsf{Gh}_{\mathcal{X}}(\Omega^i_{\mathcal{X}}(A),B) \longrightarrow \mathcal{T}(\Omega^i_{\mathcal{X}}(A),B) \longrightarrow \mathsf{Hom}\big(\mathsf{H}(\Omega^i_{\mathcal{X}}(A)),\mathsf{H}(B)\big) \longrightarrow \mathsf{Gh}_{\mathcal{X}}(\Omega^{i+1}_{\mathcal{X}}(A),B) \longrightarrow \\ \mathsf{Gh}_{\mathcal{X}}(\Omega^i_{\mathcal{X}}(A),B[1]) \longrightarrow \mathsf{Ext}^{i+1}\big(\mathsf{H}(A),\mathsf{H}(B)\big) \longrightarrow \mathcal{O}_{\Omega^{i+1}_{\mathcal{X}}(A),B} \longrightarrow 0 \end{aligned} \tag{3.4}$$

Proof. Applying the functor H to the triangles $\Omega^i_{\mathcal{X}}(A) \longrightarrow X^{i-2}_A \longrightarrow \Omega^{i-1}_{\mathcal{X}}(A) \longrightarrow \Omega^i(A)[1]$, $i \geqslant 0$, and using the vanishing conditions of the statement, we see easily that we have $\mathcal{T}(\mathcal{X},\Omega^k_{\mathcal{X}}(A)[-t+k]) = 0$, $1 \leqslant k \leqslant t-1$. As consequence we have short exact sequences $0 \longrightarrow \mathsf{H}(\Omega^k_{\mathcal{X}}(A)) \longrightarrow \mathsf{H}(X^{k-1}_A) \longrightarrow \mathsf{H}(\Omega^{k-1}_{\mathcal{X}}(A)) \longrightarrow 0$, $1 \leqslant k \leqslant t$, so $\Omega^k\mathsf{H}(A) \cong \mathsf{H}(\Omega^k_{\mathcal{X}}(A))$, $1 \leqslant k \leqslant t$. Since $\mathcal{T}(\mathcal{X},\Omega^k_{\mathcal{X}}(A)[-1]) = 0$, $1 \leqslant k \leqslant t-1$, the existence of (3.4) follows from Proposition 3.8, by replacing A with $\Omega^i_{\mathcal{X}}(A)$. $\square$

Lemma 3.11. *For any objects $A,B \in \mathcal{T}$ and any $k \geqslant 0$, there is an epimorphism:*

$$\varphi^k_{A,B} \ :\ \mathsf{Gh}_{\mathcal{X}}(\Omega^k_{\mathcal{X}}(A),B[1]) \longrightarrow \mathsf{Gh}^{[k+1]}_{\mathcal{X}}(A,B[k+1]) \longrightarrow 0$$

which is an isomorphism for $0 \leqslant k \leqslant n$, if $\mathcal{T}(\mathcal{X},B[i]) = 0$, $1 \leqslant i \leqslant n$. In this case we have a isomorphisms

$$\varphi^k_{A,B} \ :\ \mathcal{T}(\Omega^k_{\mathcal{X}}(A),B[1]) \xrightarrow{\cong} \mathcal{T}(A,B[k+1]) \quad \textit{and} \quad \varphi^n_{A,B} \ :\ \mathcal{T}(\Omega^n_{\mathcal{X}}(A),B[1]) \xrightarrow{\cong} \mathsf{Gh}^{n+1}_{\mathcal{X}}(A,B[n+1])$$

for $1 \leqslant k \leqslant n$. In particular we have a monomorphism $0 \longrightarrow \mathcal{T}(\Omega^n_{\mathcal{X}}(A),B[1]) \longrightarrow \mathcal{T}(A,B[n+1])$.

Proof. Define $\varphi^k_{A,B}(\alpha) = \omega^{k-1}_A \circ \alpha[k]$. Since ω^{k-1}_A lies in $\mathsf{Gh}^k_{\mathcal{X}}(A,\Omega^k_{\mathcal{X}}(A)[k])$ and since $\alpha[k]$ is $\mathcal{X}[k]$-ghost, it follows that $\varphi^k_{A,B}(\alpha)$ lies in $\mathsf{Gh}^{k+1}_{\mathcal{X}}(A,B[k+1])$. Now if $\beta : A \longrightarrow B[k+1]$ lies in $\mathsf{Gh}^{k+1}_{\mathcal{X}}(A,B[k+1])$, by the Ghost Lemma, there exists a map $\gamma : \Omega^{k+1}_{\mathcal{X}}(A) \longrightarrow B[k+1]$ such that $\beta = \omega^k_A \circ \gamma = \omega^{k-1}_A \circ h^k_A[k] \circ \gamma$. Clearly the map $h^k_A \circ \gamma[-k] : \Omega^k_{\mathcal{X}}(A) \longrightarrow B[1]$ is $\mathcal{X}$-ghost and $\varphi^k_{A,B}(h^k_A \circ \gamma[-k]) = \omega^{k-1}_A \circ h^k_A[k] \circ \gamma = \omega^k_A \circ \gamma = \beta$, so $\varphi^k_{A,B}(\alpha)$ is surjective. Assume now that $\mathcal{T}(\mathcal{X},B[i]) = 0$, $1 \leqslant i \leqslant n$ and let $\alpha \in \mathsf{Ker}\,\varphi^k_{A,B}$, where $1 \leqslant k \leqslant n$. Then $\omega^{k-1}_A \circ \alpha[k] = 0$ and therefore $\alpha[k]$ factorizes through the cone $\mathsf{Cell}_{k-1}(A)[1]$ of ω^{k-1}_A. Since $\mathsf{Cell}_{k-1}(A)[1]$ lies in $(\mathcal{X}\star\mathcal{X}[1]\star\cdots\mathcal{X}[k-1])[1] = \mathcal{X}[1]\star\cdots\star\mathcal{X}[k]$, it is easy to see that the condition $\mathcal{T}(\mathcal{X},B[i]) = 0$, $1 \leqslant i \leqslant k$ forces any map from an object in $\mathcal{X}[1]\star\cdots\star\mathcal{X}[k]$ to $B[k+1]$ to be zero. Hence $\alpha[k]$ or equivalently α is zero. We infer that $\varphi^k_{A,B}$ is injective for $0 \leqslant k \leqslant n$. Now observe that, since $\mathcal{T}(\mathcal{X}, B[1]) = 0$, we have $\mathsf{Gh}_{\mathcal{X}}(\Omega^k_{\mathcal{X}}(A), B[1]) = \mathcal{T}(\Omega^k_{\mathcal{X}}(A), B[1])$, $\forall k \geqslant 0$. We show by induction on k that $\mathsf{Gh}^{[k]}_{\mathcal{X}}(A, B[k]) = \mathcal{T}(A, B[k])$, $1 \leqslant k \leqslant n$. Indeed if $k = 1$, then any map $\alpha : A \longrightarrow B[1]$ is $\mathcal{X}$-ghost since $\mathcal{T}(\mathcal{X}, B[1]) = 0$. If $k = 2$ and $\alpha : A \longrightarrow B[2]$ is a map, then since $\mathcal{T}(\mathcal{X}, B[2]) = 0$, the composition $X^0_A \longrightarrow A \longrightarrow B[2]$ is zero and therefore $\alpha = h^0_A \circ \alpha_1$ for some map $\alpha_1 : \Omega^1_{\mathcal{X}}(A)[1] \longrightarrow B[2]$. Clearly α_1 is $\mathcal{X}[1]$-ghost since $\mathcal{T}(\mathcal{X}, B[1]) = 0$. Hence α lies in $\mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[2])$ and therefore $\mathsf{Gh}^{[2]}_{\mathcal{X}}(A, B[2]) = \mathcal{T}(A, B[2])$. Continuing in this way and using that $k \leqslant n$, we see that any map $\alpha : A \longrightarrow B[k+1]$ admits a factorization $\alpha = \omega^k_A \circ \alpha_{k+1}[k]$ for some map $\alpha_{k+1} : \Omega^{k+1}_{\mathcal{X}}(A)[1] \longrightarrow B[1]$. It follows that α lies in $\mathsf{Gh}^{[k+1]}_{\mathcal{X}}(A, B[k+1])$ and therefore $\mathsf{Gh}^{[k+1]}_{\mathcal{X}}(A, B[k+1]) = \mathcal{T}(A, B[k+1]$ for $0 \leqslant k \leqslant n$. □

Corollary 3.12. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$. Let $t \geqslant 2$ and assume that $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant t-1$, and $\mathcal{T}(\mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0$, $1 \leqslant i \leqslant t$. Let $A \in \mathcal{T}$ be such that: $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$ and $\mathcal{T}(\mathcal{X}, A[-i]) = 0$, $1 \leqslant i \leqslant t$. Then $\forall B \in \mathcal{T}$, we have $\mathsf{Ext}^{t+1}(\mathsf{H}(A), \mathsf{H}(B)) = 0$, so $\mathsf{pd}\, \mathsf{H}(A) \leqslant t$, and:*

$$\mathsf{Ext}^t\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\cong} \mathsf{Gh}_{\mathcal{X}}(\Omega^{t-1}_{\mathcal{X}}(A), B[1]) \quad \textit{and} \quad \mathsf{Ext}^{t-1}\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\cong} \frac{\mathsf{Gh}_{\mathcal{X}}(\Omega^{t-2}_{\mathcal{X}}(A), B[1])}{\mathsf{Gh}^{[2]}_{\mathcal{X}}(\Omega^{t-2}_{\mathcal{X}}(A), B[1])}$$

If in addition $\mathcal{T}(\mathcal{X}, B[1]) = \cdots = \mathcal{T}(\mathcal{X}, B[t-1]) = 0$, then: $\mathsf{Ext}^t\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\cong} \mathsf{Gh}^{[t]}_{\mathcal{X}}(A, B[t])$.

Proof. The assumptions on A imply easily that $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$, see Proposition 4.3(iii) below. Then $\Omega^{t-1}_{\mathcal{X}}(A) \in \mathcal{X} \star \mathcal{X}[1]$, and therefore we have $\mathcal{O}_{\Omega^k_{\mathcal{X}}(A), -} = 0$, for $k = t, t-1$, and $\mathsf{Gh}_{\mathcal{X}}(\Omega^t_{\mathcal{X}}(A), -) = 0$. It follows from (3.4), for $i = t$, that $\mathsf{Ext}^{t+1}(\mathsf{H}(A), \mathsf{H}(B) = 0$. Next setting $i = t-1$ in (3.4), we have the first isomorphism, and setting $i = t-2$ in (3.4) we have the second isomorphism. Finally if $\mathcal{T}(\mathcal{X}, B[i]) = 0$, $1 \leqslant i \leqslant t-1$, Corollary 3.11 shows that $\mathsf{Gh}_{\mathcal{X}}(\Omega^{t-1}_{\mathcal{X}}(A), B[1]) \cong \mathsf{Gh}^{[t]}_{\mathcal{X}}(A, B[t]$ and therefore the last isomorphism follows. □

4. Rigid Subcategories

Throughout this section we fix a triangulated category $\mathcal{T}$ and a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{T}$ which is closed under direct summands and isomorphisms.

For $n \geqslant 1$, we consider the following subcategories associated to $\mathcal{X}$:

$$\mathcal{X}^{\top}_n := \big\{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[i]) = 0,\ 1 \leqslant i \leqslant n\big\} \quad \text{and} \quad {}^{\top}_{n}\mathcal{X} := \big\{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0,\ 1 \leqslant i \leqslant n\big\}$$

We also set: $\mathcal{X}^{\top} := \mathcal{X}^{\top}_0 = \{A \in \mathcal{T} \,|\, \mathcal{T}(\mathcal{X}, A) = 0\}$ and ${}^{\top}\mathcal{X} := {}^{\top}_0\mathcal{X} = \{A \in \mathcal{T} \,|\, \mathcal{T}(A, \mathcal{X}) = 0\}$.

Observe that $\mathcal{X}^{\top} = \mathcal{X}^{\top}_0 = \mathcal{X}^{\top}_1[1]$ and ${}^{\top}\mathcal{X} = {}^{\top}_0\mathcal{X} = ({}^{\top}_1\mathcal{X})[-1]$, and we have filtrations:

$$\mathcal{T} \supseteq \mathcal{X}^{\top}_1 \supseteq \mathcal{X}^{\top}_2 \supseteq \cdots \supseteq \mathcal{X}^{\top}_n \supseteq \cdots \tag{4.1}$$

$$\mathcal{X} \subseteq \mathcal{X} \star \mathcal{X}[1] \subseteq \cdots \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n] \subseteq \cdots \subseteq \mathcal{T} \tag{4.2}$$

Definition 4.1. A full subcategory $\mathcal{X} \subseteq \mathcal{T}$ is called ***n*-rigid**, $n \geqslant 1$, if: $\mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0$, $1 \leqslant i \leqslant n$.

It follows that that $\mathcal{X}$ is n-rigid if and only if $\mathcal{X} \subseteq \mathcal{X}^{\top}_n$ or equivalently $\mathcal{X} \subseteq {}^{\top}_n\mathcal{X}$. We show that that contravariantly finite n-rigid subcategories in $\mathcal{T}$ give rise to torsion pairs in the sense of the following definition, see [12], [16]. A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{T}$ is called a **torsion pair** in $\mathcal{T}$, if $\mathcal{T}(\mathcal{X}, \mathcal{Y}) = 0$ and for any object A in $\mathcal{T}$ there is a triangle $X_A \longrightarrow A \longrightarrow Y^A \longrightarrow X_A[1]$, where $X_A \in \mathcal{X}$ and $Y^A \in \mathcal{Y}$. A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of subcategories of $\mathcal{T}$ is called a **torsion triple** if $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs.

If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{T}$, then clearly the map $X_A \longrightarrow A$ is a right $\mathcal{X}$-approximation of A, and the map $A \longrightarrow Y^A$ is a left $\mathcal{Y}$-approximation of A. Hence $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite in $\mathcal{T}$. Moreover it is easy to see that $\mathcal{X}^{\top} = \mathcal{Y}$ and ${}^{\top}\mathcal{Y} = \mathcal{X}$, and $\mathcal{X} \cap \mathcal{Y} = 0$.

Proposition 4.2. *If $\mathcal{X}$ is n-rigid, $n \geqslant 1$, then we have the following.*

(i) $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_1$, *$t \geqslant 1$, and* $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_t$, *$1 \leqslant t \leqslant n$.*

(ii) $\mathsf{H}\big(\mathcal{X}[1] \star \mathcal{X}[2] \cdots \star \mathcal{X}[n]\big) = 0$ *and there is a torsion pair*

$$\big(\mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t-1],\ \mathcal{X}^{\top}_t[t]\big), \qquad 1 \leqslant t \leqslant n \tag{$*$}$$

The map $\gamma^{t-1}_A : \mathsf{Cell}_{t-1}(A) \longrightarrow A$ is a right $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-1])$-approximation of A, the map $\mathsf{H}(\gamma^{t-1}_A)$ is invertible, and the map $\omega^{t-1}_A : A \longrightarrow \Omega^t_{\mathcal{X}}(A)[t]$ is a left $\mathcal{X}^{\top}_t[t]$-approximation of A.

(iii) $\forall t = 1, 2, \cdots, n$: $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$ *if and only if* $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$.

Proof. (i) Consider the triangles associated to A constructed before:

$$\Omega^t_{\mathcal{X}}(A) \longrightarrow X^{t-1}_A \longrightarrow \Omega^{t-1}_{\mathcal{X}}(A) \longrightarrow \Omega^t_{\mathcal{X}}(A)[1] \qquad (T^t_A)$$

Applying the homological functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ to the triangles (T^t_A) and using that $\mathcal{X}$ is n-rigid, it follows that $\mathsf{H}(\Omega^t_{\mathcal{X}}(A)[1]) = 0$, $\forall t \geqslant 1$, and $\mathsf{H}(\Omega^t_{\mathcal{X}}(A)[i]) \cong \mathsf{H}(\Omega^{t+1}_{\mathcal{X}}(A)[i+1])$, for $1 \leqslant t \leqslant n-1$ and $1 \leqslant i \leqslant n-1$. Hence for any $t = 1, 2, \cdots, n$, we have: $\mathsf{H}(\Omega^t_{\mathcal{X}}(A)[t]) \cong \mathsf{H}(\Omega^{t-1}_{\mathcal{X}}(A)[t-1]) \cong \cdots \cong \mathsf{H}(\Omega^2_{\mathcal{X}}(A)[2]) \cong \mathsf{H}(\Omega^1_{\mathcal{X}}(A)[1]) = 0$. It follows that $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_t$, $1 \leqslant t \leqslant n$, and $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_1$, $\forall t \geqslant 1$.

(ii), (iii) Since $\forall A \in \mathcal{T}$ we have a triangle, where $\mathsf{Cell}_{t-1}(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-1]$ and $\Omega^t_{\mathcal{X}}(A)[t] \in \mathcal{X}^{\top}_t[t]$,

$$\mathsf{Cell}_{t-1}(A) \xrightarrow{\gamma^{t-1}_A} A \xrightarrow{\omega^{t-1}_A} \Omega^t_{\mathcal{X}}(A)[t] \xrightarrow{\beta^{t-1}_A[1]} \mathsf{Cell}_{t-1}(A)[1] \qquad (\dagger)$$

it suffices to show that $\mathcal{T}(A, B) = 0$, for any object A in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-1]$ and any object $B \in \mathcal{X}^{\top}_t[t]$. We have $B = C[t]$, where $C \in \mathcal{X}^{\top}_t$, i.e. $\mathcal{T}(\mathcal{X}, C[1]) = \cdots = \mathcal{T}(\mathcal{X}, C[t]) = 0$. On the other hand, since $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-1]$, there are triangles $X_i[i] \longrightarrow A_i \xrightarrow{\alpha_{i+1}} A_{i+1} \longrightarrow X_i[i+1]$, for $i = 0, 1, \cdots, t-2$, where $A_0 = A$ and $X_i \in \mathcal{X}$, $A_{t-1} \in \mathcal{X}$. Let $f : A \longrightarrow C[t]$ be a map. Then the composition $X_0 \longrightarrow A \longrightarrow C[t]$ is zero and therefore f factors through $\alpha_1 : A \longrightarrow A_1$, say via a map $f_1 : A_1 \longrightarrow C[t]$: $f = \alpha_1 \circ f_1$. Similarly the composition $X_1[1] \longrightarrow A_1 \longrightarrow C[t]$ is zero, hence f_1 factors through $\alpha_2 : A_1 \longrightarrow A_2$, say via a map $f_2 : A_2 \longrightarrow C[t]$: $f_1 = \alpha_2 \circ f_2$. Continuing in this way we deduce after $t-2$ steps that the map $f_{t-3} : A_{t-2} \longrightarrow C[t]$ admits a factorization $f_{t-3} = \alpha_{t-2} \circ f_{t-2}$. Since the composition $X_{t-2}[t-2] \longrightarrow A_{t-2} \longrightarrow C[t]$ is zero, the map f_{t-2} factorizes through $\alpha_{t-1} : A_{t-2} \longrightarrow A_{t-1}$, say via a map $f_{t-1} : A_{t-1} \longrightarrow C[t]$: $f_{t-2} = \alpha_{t-1} \circ f_{t-1}$. Since $A_{t-1} \in \mathcal{X}[t-1]$, so $A_{t-1} = X^*[t-1]$, with $X^* \in \mathcal{X}$, the map f_{t-1} is zero since it lies in $\mathcal{T}(A_{t-1}, C[t]) \cong \mathcal{T}(X^*, C[1]) = 0$. Then $f = \alpha_1 \circ f_1 = \cdots = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{t-1} \circ f_{t-1} = 0$. Hence $(*)$ is a torsion pair in $\mathcal{T}$, and it remains to show that $\mathsf{H}(\gamma^t_A)$ is invertible, $1 \leqslant t \leqslant n-1$. This follows directly from (i) and the construction of the cellular tower $(C^{\bullet}_A)$ of A.

(iv) It follows from the triangle $(\dagger)$ that $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$ implies that $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$. Conversely assume that $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$. Then the right $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t])$-approximation $\gamma^t_A : \mathsf{Cell}_t(A) \longrightarrow A$ splits and therefore $\omega^t_A = \omega^{t-1}_A \circ h^t_A[t] = 0$. Hence the map $h^t_A[t]$ factorizes through the cone $\mathsf{Cell}^{t-1}_A[1]$ of ω^{t-1}_A, say via a map $f : \mathsf{Cell}_{t-1}(A)[1] \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t+1]$. We show that $f = 0$ or equivalently $f[-1] = 0$. Indeed there are triangles $X_i[i] \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow X_i[1+1]$, where the X_i lie in $\mathcal{X}$, $M_0 = \mathsf{Cell}_{t-1}(A)$, $M_i \in \mathcal{X}[i] \star \cdots \mathcal{X}[t-1]$, and $M_{t-1} = X_{t-1}[t-1] \in \mathcal{X}$. Now $\Omega^{t+1}_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_t$. Indeed this follows form (i) if $t \leqslant n-1$ and by applying H to the triangle $\Omega^{n+1}_{\mathcal{X}}(A) \longrightarrow X^n_A \longrightarrow \Omega^n_{\mathcal{X}}(A) \longrightarrow \Omega^{n+1}_{\mathcal{X}}(A)[1]$ and using that $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_n$, if $t = n$. It follows that the composition $X_0 \longrightarrow M_0 \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ is zero, hence $f[-1] : M_0 \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ factorizes through $M_0 \longrightarrow M_1$. Similarly since the composition $X_1[1] \longrightarrow M_1 \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ is zero, the map $M_1 \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ factorizes through $M_1 \longrightarrow M_2$, hence $f[-1]$ factorizes through $M_0 \longrightarrow M_2$. Continuing in this way we find that $M_{t-3} \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ factorizes through M_{t-2}. Finally since the map $X_{t-2}[t-2] \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ is zero, the map $M_{t-2}[t-2] \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$ factorizes through $M_{t-1} = X_{t-1}[t-1] \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$. This last map being zero, it follows that so is the map $f[-1] : M_0 \longrightarrow \Omega^{t+1}_{\mathcal{X}}(A)[t]$. Hence $f = 0$ and therefore $h^t_A[t] = 0$. Then from the triangles, $X^t_A[t] \longrightarrow \Omega^t_{\mathcal{X}}(A)[t] \xrightarrow{h^t_A[t]} \Omega^{t+1}_{\mathcal{X}}(A)[t+1] \longrightarrow X^t_A[t+1]$, cf. Remark 2.2, we deduce that $\Omega^t_{\mathcal{X}}(A)[t]$ lies in $\mathcal{X}[t]$ as a direct summand of $X^t_A[t]$. It follows that $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$. $\square$

As a direct consequence of the Ghost Lemma (Proposition 3.2) and Proposition 4.2 we have the following.

Corollary 4.3. *Let $\mathcal{X}$ be n-rigid. Then for any object $A \in \mathcal{T}$ and $1 \leqslant t \leqslant n$, the following are equivalent:*

(i) $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$.
(ii) $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$.
(iii) $\mathsf{Gh}^{[t+1]}_{\mathcal{X}}(A, -) = 0$.

Combining Proposition 4.2(ii) and Remark 2.2, we also have the following.

Corollary 4.4. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$ and $A \in \mathcal{T}$. Then the maps $\gamma^t_A : \mathsf{Cell}_t(A) \longrightarrow A$ and $\alpha^t_A : \mathsf{Cell}_{t-1}(A) \longrightarrow \mathsf{Cell}_t(A)$ of the cellular tower of A induce isomorphisms:*

$$\mathsf{H}(A) \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_1(A)) \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_2(A)) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_n(A))$$

and exact sequences:

$$\mathsf{H}(X^1_A) \longrightarrow \mathsf{H}(\mathsf{Cell}_0(A)) \longrightarrow \mathsf{H}(\mathsf{Cell}_1(A)) \longrightarrow 0$$
$$0 \longrightarrow \mathsf{H}(\mathsf{Cell}_n(A)) \longrightarrow \mathsf{H}(\mathsf{Cell}_{n+1}(A)) \longrightarrow \mathsf{H}(X^{n+1}_A[n+1])$$

The following gives a condition ensuring that the filtrations (3.2), (4.1) and (4.2) stabilize:

Theorem 4.5. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$. Then the following are equivalent:*

(i) $\mathcal{X}_n^{\top} = \mathcal{X}$.
(ii) $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$.
(iii) $\mathsf{Gh}^{[n+1]}_{\mathcal{X}}(\mathcal{T}) = 0$.
(iv) $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.

If one of the above equivalent conditions holds, then $\mathcal{X}^{\top}_{n+k} = 0$, $\forall k \geqslant 1$.

Proof. Since by Proposition 4.2, $\Omega^n_{\mathcal{X}}(A)[n]$ lies in $\mathcal{X}^{\top}_n[n]$, condition (i) implies that $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$. Hence (i) $\Rightarrow$ (ii), and by Corollary 4.3 we have (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). We show that (iv) $\Rightarrow$ (i). By Proposition 4.2 we know that, for any object $A \in \mathcal{T}$, the map $\omega^{n-1}_A : A \longrightarrow \Omega^n_{\mathcal{X}}(A)[n]$ is a left $\mathcal{X}^{\top}_n$-approximation of A and $\Omega^n_{\mathcal{X}}(A)[n]$ lies in $\mathcal{X}[n]$ since $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$ by Corollary 4.3. Hence if $A \in \mathcal{X}^{\top}_n[n]$, the map ω^{n-1}_A is split monic and therefore A lies in $\mathcal{X}[n]$ as a direct summand of $\Omega^n_{\mathcal{X}}(A)[n]$. We infer that $\mathcal{X}^{\top}_n[n] \subseteq \mathcal{X}[n]$, or equivalently $\mathcal{X}^{\top}_n \subseteq \mathcal{X}$. Since $\mathcal{X}$ is n-rigid, we have $\mathcal{X} \subseteq \mathcal{X}^{\top}_n$, hence $\mathcal{X}^{\top}_n = \mathcal{X}$, i.e. (iv) $\Rightarrow$ (i).

If one of the first four equivalent conditions holds, then let $A \in \mathcal{X}^{\top}_{n+1}$ and $B = A[1]$. Then $\mathcal{T}(\mathcal{X}, B) = 0$ and clearly $B \in \mathcal{X}^{\top}_n$. Since $\mathcal{X}^{\top}_n = \mathcal{X}$, it follows that $\mathcal{T}(B, B) = 0$, i.e. $B = A[1] = 0$ and therefore $A = 0$. Hence $\mathcal{X}^{\top}_{n+1} = 0$ and then trivially $\mathcal{X}^{\top}_{n+k} = 0$, $\forall k \geqslant 1$. □

Under the equivalent conditions of Theorem 4.6, we now show that $\mathsf{Ker}\,\mathsf{H} = \mathcal{X}^{\top}$ admits a nice description.

Corollary 4.6. *Let $\mathcal{X}$ be a n-rigid subcategory of $\mathcal{T}$, where $n \geqslant 1$. If $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$, then the full subcategories $\mathcal{X}^{\top}$ and ${}^{\top}\mathcal{X}$ are functorially finite in $\mathcal{T}$. Moreover:*

$$\mathcal{X}^{\top} = \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n] \qquad \text{and} \qquad {}^{\top}\mathcal{X} = \mathcal{X}[-n] \star \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$$

Proof. By Remark 2.2 we have $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t]$, $0 \leqslant t \leqslant n$. Hence $\Omega^1_{\mathcal{X}}(A)[1] \in (\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1])[1] = \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. By Proposition 4.2(iii) we have $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n] \subseteq \mathcal{X}^{\top}$. Consider the triangle $\Omega^1_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. If $A \longrightarrow B$ is a map, where $B \in \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$, then the composition $X^0_A \longrightarrow A \longrightarrow B$ is zero and therefore $A \longrightarrow B$ factorizes through $h^1_A : A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. Hence h^1_A is a left $(\mathcal{X}[1] \star \cdots \star \mathcal{X}[n])$-approximation of A. If $A \in \mathcal{X}^{\top}$, then the map $X^0_A \longrightarrow A$ is zero and therefore A lies in $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ as a direct summand of $\Omega^1_{\mathcal{X}}(A)[1]$, i.e. $\mathcal{X}^{\top} \subseteq \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. By Proposition 4.4, $\mathcal{X} \star \cdots \star \mathcal{X}[n-1]$ is contravariantly finite. This clearly implies that $(\mathcal{X} \star \cdots \star \mathcal{X}[n-1])[1] = \mathcal{X}[1] \star \cdots \star \mathcal{X}[n] = \mathcal{X}^{\top}$ is also contravariantly finite. The proof for ${}^{\top}\mathcal{X}$ is dual and is left to the reader. □

We close this section with the following vanishing result, which, will be useful later and, gives conditions ensuring that the sequence of cones $\big\{\omega^n_A : A \longrightarrow \Omega^n_{\mathcal{X}}(A)[n]\big\}_{n \geqslant 0}$ is eventually trivial.

Lemma 4.7. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$. Then $\forall A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$:*

$$\mathcal{T}(\Omega^t_{\mathcal{X}}(A), \mathcal{X}[n-t+1]) = 0, \;\; 1 \leqslant t \leqslant n \tag{4.3}$$

Moreover $\omega^t_A = 0$, $\forall t \geqslant n$, and there is a decomposition: $\mathsf{Cell}_t(A) \cong A \oplus \Omega^{t+1}_{\mathcal{X}}(A)[t]$*, $\forall t \geqslant n$.*

Proof. Since $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$, by Corollary 4.3 we have $\Omega^n_{\mathcal{X}}(A)$ lies in $\mathcal{X}$. Then the triangles

$$\Omega^t_{\mathcal{X}}(A) \longrightarrow X^{t-1}_A \longrightarrow \Omega^{t-1}_{\mathcal{X}}(A) \longrightarrow \Omega^t_{\mathcal{X}}(A)[1], \quad 1 \leqslant t \leqslant n \tag{T^t_A}$$

show that $\Omega^t_{\mathcal{X}}(A)$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t]$. Since by Proposition 4.2 we have a torsion pair $\big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t], \mathcal{X}^{\top}_{n-t+1}[n-t+1]\big)$ in $\mathcal{T}$ and $\mathcal{X}[n-t+1] \subseteq \mathcal{X}^{\top}_{n-t+1}[n-t+1]$, because $\mathcal{X} \subseteq \mathcal{X}^{\top}_{n-t+1}$, we get directly (4.3). On the other hand the right $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n])$-approximation γ^n_A of A splits and therefore $\omega^n_A = 0$. Then from the tower of triangles $(C^{\bullet}_A)$, we have $\omega^t_A = 0$, $\forall t \geqslant n$. □

5. Homological Dimension

Let $\mathcal{T}$ be a triangulated category with split idempotents and $\mathcal{X}$ a contravariantly finite subcategory of $\mathcal{T}$.

5.1. **The adjoint pair** $(\Sigma_{\mathcal{X}}, \Omega_{\mathcal{X}})$**.** Let A be in $\mathcal{T}$ and consider triangles

$$\Omega_{\mathcal{X}}(A) \xrightarrow{g_A^0} X_A^0 \xrightarrow{f_A^0} A \xrightarrow{h_A^0} \Omega_{\mathcal{X}}(A)[1] \quad \text{and} \quad \tilde{\Omega}_{\mathcal{X}}(A) \xrightarrow{\tilde{g}_A^0} \tilde{X}_A^0 \xrightarrow{\tilde{f}_A^0} A \xrightarrow{\tilde{h}_A^0} \tilde{\Omega}_{\mathcal{X}}(A)[1]$$

where the maps $X_A^0 \longrightarrow A \longleftarrow \tilde{X}_A^0$ are right $\mathcal{X}$-approximations. Then there are maps $\beta : X_A^0 \longrightarrow \tilde{X}_A^0$ and $\tilde{\beta} : \tilde{X}_A^0 \longrightarrow X_A^0$ inducing morphisms of triangles:

$$\begin{tikzcd}
\Omega_{\mathcal{X}}(A) \arrow[r, "g_A^0"] \arrow[d, "\gamma"] & X_A^0 \arrow[r, "f_A^0"] \arrow[d, "\beta"] & A \arrow[r, "h_A^0"] \arrow[d, "\parallel"] & \Omega_X(A)[1] \arrow[d, "\gamma[1]"] \\
\tilde{\Omega}_{\mathcal{X}}(A) \arrow[r, "\tilde{g}_A^0"] & \tilde{X}_A^0 \arrow[r, "\tilde{f}_A^0"] & A \arrow[r, "\tilde{h}_A^0"] & \tilde{\Omega}_{\mathcal{X}}(A)[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\Omega}_{\mathcal{X}}(A) \arrow[r, "\tilde{g}_A^0"] \arrow[d, "\tilde{\gamma}"] & \tilde{X}_A^0 \arrow[r, "\tilde{f}_A^0"] \arrow[d, "\tilde{\beta}"] & A \arrow[r, "\tilde{h}_A^0"] \arrow[d, "\parallel"] & \tilde{\Omega}_X(A)[1] \arrow[d, "\tilde{\gamma}[1]"] \\
\Omega_{\mathcal{X}}(A) \arrow[r, "g_A^0"] & X_A^0 \arrow[r, "f_A^0"] & A \arrow[r, "h_A^0"] & \Omega_{\mathcal{X}}(A)[1]
\end{tikzcd}$$

Since $h_A^0[-1] \circ \gamma \circ \tilde{\gamma} = h_A^0[-1]$ and $\tilde{h}_A^0 \circ \tilde{\gamma} \circ \gamma = \tilde{h}_A^0[-1]$, there are maps $\kappa : X_A^0 \longrightarrow \Omega_{\mathcal{X}}(A)$ and $\lambda : \tilde{X}_{\mathscr{A}}^0 \longrightarrow \tilde{\Omega}_{\mathcal{X}}(A)$ such that: $1_{\Omega_{\mathcal{X}}(A)} - \gamma \circ \tilde{\gamma} = g_A^0 \circ \kappa$ and $1_{\tilde{\Omega}_{\mathcal{X}}(A)} - \tilde{\gamma} \circ \gamma = \tilde{g}_A^0 \circ \lambda$. This means that the map $\underline{\gamma}$ is invertible in $\mathcal{T}/\mathcal{X}$ and $\underline{\gamma}^{-1} = \underline{\tilde{\gamma}}$. We infer that the object $\Omega_{\mathcal{X}}(A)$ is uniquely determined by A up to an isomorphism in the stable category $\mathcal{T}/\mathcal{X}$ and does not depends on the choice of the right $\mathcal{X}$-approximations. Now if $\alpha : A \longrightarrow B$ is a map in $\mathcal{T}$, then α induces a morphism of triangles indicated in the left of the following display

$$\begin{tikzcd}
\Omega_{\mathcal{X}}(A) \arrow[r, "g_A^0"] \arrow[d, "\gamma"] & X_A^0 \arrow[r, "f_A^0"] \arrow[d, "\beta"] & A \arrow[r, "h_A^0"] \arrow[d, "\alpha"] & \Omega_X(A)[1] \arrow[d, "\gamma[1]"] \\
\Omega_{\mathcal{X}}(B) \arrow[r, "g_B^0"] & X_B^0 \arrow[r, "f_B^0"] & B \arrow[r, "h_B^0"] & \Omega_{\mathcal{X}}(B)[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{\mathcal{X}}(A) \arrow[r, "g_A^0"] \arrow[d, "\gamma'"] & X_A^0 \arrow[r, "f_A^0"] \arrow[d, "\beta'"] & A \arrow[r, "h_A^0"] \arrow[d, "\alpha"] & \Omega_X(A)[1] \arrow[d, "\gamma[1]"] \\
\Omega_{\mathcal{X}}(B) \arrow[r, "g_B^0"] & X_B^0 \arrow[r, "f_B^0"] & B \arrow[r, "h_B^0"] & \Omega_{\mathcal{X}}(B)[1]
\end{tikzcd}$$

The maps β and γ are not uniquely determined, so if there are maps β' and γ' making the diagram on the right of the above display a morphism of triangles, then clearly $\gamma - \gamma' = g_A^0 \circ \rho$ for some map $\rho : X_A^0 \longrightarrow \Omega_{\mathcal{X}}(B)$. Then $\underline{\gamma} = \underline{\gamma}'$ in the stable category $\mathcal{T}/\mathcal{X}$ and this unique map is denoted by $\Omega_{\mathcal{X}}(\underline{\alpha})$. It is then easy to see that the assignments $A \longmapsto \Omega_{\mathcal{X}}(A)$ and $\alpha \longmapsto \Omega_{\mathcal{X}}(\underline{\alpha})$ define an additive functor $\Omega_{\mathcal{X}} : \mathcal{T}/\mathcal{X} \longrightarrow \mathcal{T}/\mathcal{X}$. Dually if $\mathcal{X}$ is covariantly finite, then by performing the dual constructions we obtain an additive functor $\Sigma_{\mathcal{X}} : \mathcal{T}/\mathcal{X} \longrightarrow \mathcal{T}/\mathcal{X}$.

Lemma 5.1. *If $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{T}$, then we have an adjoint pair*

$$(\Sigma_{\mathcal{X}}, \Omega_{\mathcal{X}}) : \mathcal{T}/\mathcal{X} \longrightarrow \mathcal{T}/\mathcal{X}$$

and the unit $\delta : \mathsf{Id}_{\mathcal{T}/\mathcal{X}} \longrightarrow \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}$ and the counit $\varepsilon : \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}} \longrightarrow \mathsf{Id}_{\mathcal{T}/\mathcal{X}}$ induce isomorphisms:

$$\Omega_{\mathcal{X}}(\underline{\varepsilon}) : \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}\Omega_{\mathcal{X}} \xrightarrow{\cong} \Omega_{\mathcal{X}} \quad \textit{and} \quad \Sigma_{\mathcal{X}}(\underline{\delta}) : \Sigma_{\mathcal{X}} \xrightarrow{\cong} \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}$$

Proof. Using functorial finiteness of $\mathcal{X}$ we may construct triangles

$$\begin{tikzcd}
\Omega_{\mathcal{X}}(A) \arrow[r, "f_0^{\Omega_{\mathcal{X}}(A)}"] \arrow[d, "\parallel"] & X_0^{\Omega_{\mathcal{X}}(A)} \arrow[r, "g_0^{\Omega_{\mathcal{X}}(A)}"] \arrow[d, "\rho"] & \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(A) \arrow[r, "h_0^{\Omega_{\mathcal{X}}(A)}"] \arrow[d, "\varepsilon_A"] & \Omega_{\mathcal{X}}(A)[1] \arrow[d, "\parallel"] \\
\Omega_{\mathcal{X}}(A) \arrow[r, "g_A^0"] & X_A^0 \arrow[r, "f_A^0"] & A \arrow[r, "h_A^0"] & \Omega_{\mathcal{X}}(A)[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "g_0^A"] \arrow[d, "\delta_A"] & X_0^A \arrow[r, "f_0^A"] \arrow[d, "\sigma"] & \Sigma_{\mathcal{X}}(A) \arrow[r, "h_0^A"] \arrow[d, "\parallel"] & A[1] \arrow[d, "\delta_A[1]"] \\
\Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(A) \arrow[r, "g_{\Sigma_{\mathcal{X}}(A)}^0"] & X_{\Sigma_{\mathcal{X}}(A)}^0 \arrow[r, "f_{\Sigma_{\mathcal{X}}(A)}^0"] & \Sigma_{\mathcal{X}}(A) \arrow[r, "h_{\Sigma_{\mathcal{X}}(A)}^0"] & \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(A)[1]
\end{tikzcd}$$

We leave to the reader to check that the induced maps $\underline{\varepsilon}_A : \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(\underline{A}) \longrightarrow \underline{A}$ and $\underline{\delta}_A : \underline{A} \longrightarrow \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(\underline{A})$ are natural and define the counit and the unit of an adjoint pair $(\Sigma_{\mathcal{X}}, \Omega_{\mathcal{X}})$ in $\mathcal{T}/\mathcal{X}$. Since by construction $\mathcal{T}(\mathcal{X}, h_{\Omega_{\mathcal{X}}(A)}^0) = \mathcal{T}(\mathcal{X}, \varepsilon_A) \circ \mathcal{T}(\mathcal{X}, h_A^0) = 0$ and $\mathcal{T}(h_{\Sigma_{\mathcal{X}}(A)}^0[-1], \mathcal{X}) = \mathcal{T}(\delta_A, \mathcal{X}) \circ \mathcal{T}(h_0^A[-1]) = 0$, it follows that the map $g_0^{\Omega_{\mathcal{X}}(A)} : X_0^{\Omega_{\mathcal{X}}(A)} \longrightarrow \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(A)$ is a right $\mathcal{X}$-approximation of $\Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(A)$ and the map $g_{\Sigma_{\mathcal{X}}(A)}^0 : \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(A) \longrightarrow X_{\Sigma_{\mathcal{X}}(A)}^0$ is a left $\mathcal{X}$-approximation of $\Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(A)$. As a consequence by the above triangles we infer that the maps $\Omega_{\mathcal{X}}(\underline{\varepsilon}_A) : \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(\underline{A}) \longrightarrow \Omega_{\mathcal{X}}(\underline{A})$ and $\Sigma_{\mathcal{X}}(\underline{\delta}_A) : \Sigma_{\mathcal{X}}(\underline{A}) \longrightarrow \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}\Omega_{\mathcal{X}}(\underline{A})$ are invertible. Hence we have natural isomorphisms: $\Omega_{\mathcal{X}}(\underline{\varepsilon}) : \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}\Omega_{\mathcal{X}} \xrightarrow{\cong} \Omega_{\mathcal{X}}$ and $\Sigma_{\mathcal{X}}(\underline{\delta}) : \Sigma_{\mathcal{X}} \xrightarrow{\cong} \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}$. □

5.2. **Homological Dimension.** If $A \in \mathcal{T}$, we define the $\mathcal{X}$**-projective dimension** $\mathsf{pd}_{\mathcal{X}} A$ of A to be the smallest $n \geqslant 0$ such that $\Omega_{\mathcal{X}}^n(\underline{A}) = 0$ in $\mathcal{T}/\mathcal{X}$ or equivalently $\Omega_{\mathcal{X}}^n(A) \in \mathcal{X}$. If $\Omega_{\mathcal{X}}^n(A) \notin \mathcal{X}$, $\forall n \geqslant 0$, then we set $\mathsf{pd}_{\mathcal{X}} A = \infty$. The $\mathcal{X}$**-global dimension** of $\mathcal{T}$ is defined by $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} = \sup\{\mathsf{pd}_{\mathcal{X}} A \mid A \in \mathcal{T}\}$. Note that the discussion in 5.1 shows that the invariants $\mathsf{pd}_{\mathcal{X}} A$ and $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T}$ are well-defined. Dually if $\mathcal{X}$ is covariantly finite, the $\mathcal{X}$**-injective dimension** $\mathsf{id}_{\mathcal{X}} A$ is defined, and in case $\mathcal{X}$ is functorially finite in $\mathcal{T}$, then the existence of the adjoint pair $(\Sigma_{\mathcal{X}}^n, \Omega_{\mathcal{X}}^n)$ in $\mathcal{T}/\mathcal{X}$, $\forall n \geqslant 0$, shows that $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} = \sup\{\mathsf{pd}_{\mathcal{X}} A \mid A \in \mathcal{T}\} = \sup\{\mathsf{id}_{\mathcal{X}} A \mid A \in \mathcal{T}\}$.

Lemma 5.2. *Let A be in $\mathcal{T}$. Then we have the following.*

(i) *If* $\mathsf{pd}_{\mathcal{X}} A = m < \infty$*, then* $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[m]$.
(ii) *If* $\mathcal{X}$ *is* n*-rigid, then for any* $m \leqslant n$ *we have:* $\mathsf{pd}_{\mathcal{X}} A \leqslant m$ *if and only if* $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[m]$.
(iii) *If* $\mathcal{X}$ *is* n*-rigid, and* $\mathsf{pd}_{\mathcal{X}} A = m < n$*, then* $A \in \mathcal{X}^{\top}_{n-m}$*, i.e.* $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leqslant i \leqslant n - m$.

Proof. Part (i) follows from Proposition 3.2 and part (ii) follows from Proposition 4.2.

(iii) Since $\mathsf{pd}_{\mathcal{X}} A = m$, it follows that $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[m]$. Then there exists a triangle $X_0 \longrightarrow A \longrightarrow B \longrightarrow X_0[1]$, where $X_0 \in \mathcal{X}$ and $B \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[m]$. It follows that $B[i] \in \mathcal{X}[i+1] \star \cdots \star \mathcal{X}[m+i]$ and this implies that for $1 \leqslant i \leqslant n - m$ we have $\mathcal{T}(\mathcal{X}, B[i]) = 0$ since $\mathcal{X}$ is n-rigid and $m < n$. Then applying $\mathcal{T}(\mathcal{X}, -)$ to the above triangle we see directly that $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leqslant i \leqslant n - m$. □

By Corollary 5.2 we have that if $\mathcal{X}$ is n-rigid, then $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} \leqslant m$ if and only if $\Omega^m_{\mathcal{X}}(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$ if and only if $\mathsf{Gh}^{[m+1]}_{\mathcal{X}}(A, -) = 0$, $\forall A \in \mathcal{T}$, if and only if $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[m]$.

Corollary 5.3. *Let* $\mathcal{T}$ *be a non-trivial triangulated category and* $\mathcal{X}$ *a contravariantly finite* n*-rigid subcategory of* $\mathcal{T}$*, where* $n \geqslant 1$*. Then* $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} \geqslant n$*, and* $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} = n$ *if and only if* $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.

Proof. We may assume that $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} = m < \infty$, and let $m < n$. Then we have that $\Omega^m_{\mathcal{X}}(A) \in \mathcal{X}$, $\forall A \in \mathcal{T}$ and therefore $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[m]$. Since $m < n$, $\mathcal{X}$ is m-rigid and then by Theorem 4.7 we have that $\mathcal{X}^{\top}_{m+1} = 0$. Since $\mathsf{pd}_{\mathcal{X}} A[t] \leqslant m$, $\forall t \geqslant 0$, it follows by Lemma 5.2 that $A[t] \in \mathcal{X}^{\top}_{n-m}$. This clearly implies that $A \in \mathcal{X}^{\top}_{m+1}$ and therefore $A = 0$. This contradiction shows that $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} \geqslant n$. Moreover by Lemma 5.2 we have $\mathsf{gl.dim}_{\mathcal{X}} \mathcal{T} = n$ if and only if $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. □

We use the above results to construct certain exact sequences in $\mathsf{mod}\text{-}\mathcal{X}$ which will be useful later on.

Theorem 5.4. *Let* $1 \leqslant t \leqslant n$ *and* $\mathcal{X}$ *be a contravariantly finite subcategory of* $\mathcal{T}$ *such that*

$$\mathcal{T}(\mathcal{X}, \mathcal{X}[i]) = 0, \quad \forall i \in [-t+1, t] \backslash \{0\} \tag{$*$}$$

(i) *For any* $A \in \mathcal{X}^{\top}_{t-1}[t] \cap \mathcal{X}^{\top}_1$ *there is a short exact sequence*

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(E_0) \longrightarrow \mathsf{H}(E_1) \longrightarrow 0$$

where $E_0, E_1 \in \mathcal{X}^{\top}_t[t+1]$.

(ii) *For any* $A \in \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\big) \cap \mathcal{X}^{\top}_t[t+1]$*, there is an exact sequence*

$$0 \longrightarrow \mathsf{H}(X^t) \longrightarrow \mathsf{H}(X^{n-1}) \longrightarrow \cdots\cdots \longrightarrow \mathsf{H}(X^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow \mathsf{H}(A) \longrightarrow 0$$

where the X^i, X_j *lie in* $\mathcal{X}$ *and* $\mathsf{pd}\,\mathsf{H}(A) \leqslant t$.

The split the proof into two steps.

Proposition 5.5. *Let* $n \geqslant 1$ *and* $\mathcal{X}$ *be a contravariantly finite* n*-rigid subcategory of* $\mathcal{T}$*. Then for any object* $A \in \mathcal{X}^{\top}_{t-1}[t]$*,* $1 \leqslant t \leqslant n$*, there exists a triangle*

$$A \longrightarrow E_0 \longrightarrow E_1 \longrightarrow A[1]$$

where $E_0, E_1 \in \mathcal{X}^{\top}_t[t+1]$ *and the sequence* $0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(E_0) \longrightarrow \mathsf{H}(E_1)$ *is exact. In particular if, in addition,* $A \in \mathcal{X}^{\top}_1$*, then there exists an exact sequence*

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(E_0) \longrightarrow \mathsf{H}(E_1) \longrightarrow 0$$

Proof. **Case** $n = 1$**:** Then condition $(*)$ reduces to $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, i.e. $\mathcal{X} \subseteq \mathcal{X}^{\top}_1$. Since, by Proposition 4.2, we have $\mathcal{T}(\mathcal{X}, \Omega^1_{\mathcal{X}}(A)[1]) = 0$, it follows that $\Omega^1_{\mathcal{X}}(A) \in \mathcal{X}^{\top}_1$, $\forall A \in \mathcal{T}$. For any $X \in \mathcal{X}$, consider the triangle

$$\Omega^1_{\mathcal{X}}(X[-1])[1] \longrightarrow \mathsf{Cell}_0(X[-1])[1] \longrightarrow X \longrightarrow \Omega^1_{\mathcal{X}}(X[-1])[2] \tag{5.1}$$

where $\mathsf{Cell}_0(X[-1]) = X^0_{X[-1]} \in \mathcal{X} \subseteq \mathcal{X}^{\top}_1$. Applying H and using that $\mathcal{X} \subseteq \mathcal{X}^{\top}_1$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(X[-1])[2]) \longrightarrow \mathsf{H}(\mathsf{Cell}_0(X[-1])[2]) \longrightarrow 0$$

Then the assertion follows by setting $E_0 = \Omega^1_{\mathcal{X}}(X[-1])$ and $E_1 = \mathsf{Cell}_0(X[-1])$.

Case $n \geqslant 2$**:** Let $A \in \mathcal{X}^{\top}_{t-1}[t]$, i.e. $\mathcal{T}(\mathcal{X}, A[-1]) = \mathcal{T}(\mathcal{X}, A[-2]) = \cdots = \mathcal{T}(\mathcal{X}, A[-t+1]) = 0$. Consider the triangle arising from the tower $\mathsf{Cell}_{A[-1]}$:

$$\Omega^t_{\mathcal{X}}(A[-1])[t-1] \longrightarrow \mathsf{Cell}_{t-1}(A[-1]) \longrightarrow A[-1] \longrightarrow \Omega^t_{\mathcal{X}}(A[-1])[t]$$

Setting $\Omega^t_{\mathcal{X}}(A[-1]) := B$ and $C = \mathsf{Cell}_{t-1}(A[-1])[1])$, we have a triangle

$$B[t] \longrightarrow C \longrightarrow A \longrightarrow B[t+1]$$

and, by Proposition 4.2, we know that $B \in \mathcal{X}_t^{\top}$ and $C \in \mathcal{X}[1] \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$. Applying H to this triangle and using that $\mathsf{H}(C) = 0$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B[t+1]) \longrightarrow \mathsf{H}(C[1])$$

where by construction we have $B[t+1] \in \mathcal{X}_t^{\top}[t+1]$. Now applying the functor H to the triangles

$$B[1] \longrightarrow C[-t+1] \longrightarrow A[-t+1] \longrightarrow B[2]$$
$$B[2] \longrightarrow C[-t+2] \longrightarrow A[-t+2] \longrightarrow B[3]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$B[t] \longrightarrow C \longrightarrow A \longrightarrow B[t+1]$$

and using that $\mathcal{T}(\mathcal{X}, A[-i]) = 0$, $1 \leqslant i \leqslant t-1$, and $\mathcal{T}(\mathcal{X}, B[j]) = 0$, $1 \leqslant j \leqslant t$, we have:

$$\mathcal{T}(\mathcal{X}, C[-t+1]) = \mathcal{T}(\mathcal{X}, C[-t+2]) = \mathcal{T}(\mathcal{X}, C[-t+3]) = \cdots = \mathcal{T}(\mathcal{X}, t[-1]) = \mathcal{T}(\mathcal{X}, C) = 0$$

This means that $C[-t] \in \mathcal{X}_t^{\top}$, and therefore $C[1] \in \mathcal{X}_t^{\top}[t+1]$. Now the assertion follows by setting $E_0 = \Omega^t_{\mathcal{X}}(A[-1]) = B$ and $E_1 = \mathsf{Cell}_{t-1}(A[-1])[2] = A[1]$. □

Proposition 5.6. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$, $n \geqslant 1$. Assume that $\mathcal{X} \subseteq \mathcal{X}_{t-1}^{\top}[t]$, if $2 \leqslant t \leqslant n$. If A lies in $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]) \cap \mathcal{X}_t^{\top}[t+1]$, then $\Omega^t_{\mathcal{X}}(A) \in \mathcal{X}$ and $\mathsf{pd}\,\mathsf{H}(A) \leqslant t$.*

Proof. If $t = 1$, then since $A \in \mathcal{X} \star \mathcal{X}[1]$, there exists a triangle $X_0 \longrightarrow A \longrightarrow X_1[1] \longrightarrow X_0[1]$, where the X_i lie in $\mathcal{X}$. Applying H and using that $\mathcal{T}(\mathcal{X}, A[-1]) = 0 = \mathcal{T}(\mathcal{X}, \mathcal{X}[1])$, it follows that the sequence $0 \longrightarrow \mathsf{H}(X_1) \longrightarrow \mathsf{H}(X_0) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ is exact. This means that $\mathsf{pd}\,\mathsf{H}(A) \leqslant 1$ and the map $X \longrightarrow A$ is a right $\mathcal{X}$-approximation of A, so its cocone $X_1 \in \mathcal{X}$ can be chosen as $\Omega^1_{\mathcal{X}}(A)$. If $2 \leqslant t \leqslant n$, then Corollary 3.12 shows that $\Omega^t_{\mathcal{X}}(A) \in \mathcal{T}$ and $\mathsf{Ext}^{t+1}(\mathsf{H}(A), \mathsf{H}(B) = 0$, $\forall B \in \mathcal{T}$. Hence $\mathsf{pd}\,\mathsf{H}(A) \leqslant t$. □

6. Cluster-Tilting Subcategories

Let as before $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a full subcategory of $\mathcal{T}$ which is closed under direct summands and isomorphisms. The results of section 5 show that if $\mathcal{X}$ is contravariantly finite and satisfies $\mathcal{X} = \mathcal{X}_n^{\top}$, then $\mathcal{X}$ enjoys special properties. In this section we give several characterizations of such subcategories.

First we observe the following symmetry.

Proposition 6.1. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$, and $n \geqslant 1$. Then the following are equivalent:*

(i) *$\mathcal{X}$ is contravariantly finite and $\mathcal{X} = \mathcal{X}_n^{\top}$.*
(ii) *$\mathcal{X}$ is contravariantly finite and both $\mathcal{X}$ and $\mathcal{X}_n^{\top}$ are n-rigid.*
(iii) *$\mathcal{X}$ is covariantly finite and $\mathcal{X} = {}^{\top}_{n}\mathcal{X}$.*
(iv) *$\mathcal{X}$ is covariantly finite and both $\mathcal{X}$ and ${}^{\top}_{n}\mathcal{X}$ are n-rigid.*

Proof. (i) ⇒ (ii) and (iii) ⇒ (iv) The proof trivial.

(ii) ⇒ (i) ⇐ (iv) Assume that (ii) holds. Since $\mathcal{X}$ is n-rigid, we have $\mathcal{X} \subseteq \mathcal{X}_n^{\top}$. Let A be an object of $\mathcal{T}$ and consider the associated tower of triangles $(C^{\bullet}_A)$. By Proposition 4.2, the map $\omega^{n-1}_A : A \longrightarrow \Omega^n_{\mathcal{X}}(A)[n]$ is a left $\mathcal{X}_n^{\top}[n]$-approximation of A. If A lies in $\mathcal{X}_n^{\top}$, then since the latter is n-rigid it follows that $\omega^{n-1}_A = 0$ and therefore ω^{n-2}_A factorizes through the left cone $X^{n-1}_A[n-1]$ of $h^{n-1}_A[n-1]$. Since both A and X^{n-1}_A lie in $\mathcal{X}_n^{\top}$ and the latter is n-rigid, it follows that $\mathcal{T}(A, X^{n-1}_A[n-1]) = 0$ and this implies that $\omega^{n-2}_A = 0$. Continuing in this way after $n-1$ steps we deduce that $\omega^1_A = 0$ and therefore the map ω^0_A factorizes through the left cone $X^1_A[1]$ of $h^1_A[1]$. Since both A and X^1_A lie in $\mathcal{X}_n^{\top}$ and the latter is n-rigid, we have $\mathcal{T}(A, X^1_A[1]) = 0$ and this implies that $h^0_A = \omega^0_A = 0$. Then A lies in $\mathcal{X}$ as a direct summand of X^0_A. Hence $\mathcal{X}_n^{\top} \subseteq \mathcal{X}$ and therefore $\mathcal{X} = \mathcal{X}_n^{\top}$. The proof that (iv) ⇒ (i) is dual to the proof of the implication (ii) ⇒ (i) using cocellular towers, cf. 2.2, and is left to the reader.

(i) ⇒ (iii) Let $\mathcal{X} = \mathcal{X}_n^{\top}$. By Proposition 4.2 we know that $\mathcal{X}_n^{\top}[n]$ is covariantly finite. Let A be in $\mathcal{T}$ and let $\omega^{n-1}_{A[n]} : A[n] \longrightarrow \Omega^n_{\mathcal{X}}(A[n])[n]$ be a left $\mathcal{X}_n^{\top}[n]$-approximation of A. Then clearly the map $\omega^{n-1}_{A[n]}[-n] : A \longrightarrow \Omega^n_{\mathcal{X}}(A[n])$ is a left $\mathcal{X}_n^{\top}$-approximation of A, so $\mathcal{X}_n^{\top}$ is covariantly finite. Since $\mathcal{X}_n^{\top} = \mathcal{X}$, the last map is a left $\mathcal{X}$-approximation of A and therefore $\mathcal{X}$ is covariantly finite in $\mathcal{T}$. Since $\mathcal{X}$ is n-rigid, we have $\mathcal{X} \subseteq {}^{\top}_{n}\mathcal{X}$. Let A be in ${}^{\top}_{n}\mathcal{X}$. Consider the map $\omega^{n-1}_A : A \longrightarrow \Omega^n_{\mathcal{X}}(A)[n]$. Then $\Omega^n_{\mathcal{X}}(A)[n] \in \mathcal{X}[n]$ and therefore $\omega^{n-1}_A = 0$ since $A \in {}^{\top}_{n}\mathcal{X}$. Hence $\omega^{n-2}_A \circ h^{n-1}_A[n-1] = 0$ and therefore ω^{n-1}_A factors through the left cone $X^{n-1}_A[n-1]$ of $h^{n-1}_A[n-1]$, say via a map $A \longrightarrow X^{n-1}_A[n-1]$. Since $A \in {}^{\top}_{n}\mathcal{X}$, the last map is zero and therefore $\omega^{n-2}_A = 0$. Continuing in this we deduce after $n-1$ steps that $\omega^1_A = 0$ and therefore $\omega^0_A \circ h^1_A[1] = 0$. Then ω^0_A factorizes through the left cone $X^1_A[1]$ of $h^1_A[1]$. However $\mathcal{T}(A, X^1_A[1]) = 0$, since $A \in {}^{\top}_{n}\mathcal{X}$. This implies that $h^0_A = \omega^0_A$ and therefore A lies in $\mathcal{X}$ as a direct summand X^0_A. Hence ${}^{\top}_{n}\mathcal{X} \subseteq \mathcal{X}$ and therefore $\mathcal{X} = {}^{\top}_{n}\mathcal{X}$. □

Definition 6.2. [21, 16] A full subcategory $\mathcal{X}$ of $\mathcal{T}$ is called a $(n+1)$-**cluster tilting**, $n \geqslant 1$, if:

(i) $\mathcal{X}$ is functorially finite.
(ii) $\mathcal{X} = \{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[i]) = 0, \ 1 \leqslant i \leqslant n\}$, i.e. $\mathcal{X} = \mathcal{X}_n^{\top}$.
(iii) $\mathcal{X} = \{A \in \mathcal{T} \mid \mathcal{T}(A, \mathcal{X}[i]) = 0, \ 1 \leqslant i \leqslant n\}$, i.e. $\mathcal{X} = {}_n^{\top}\mathcal{X}$.

Now we can prove the main result of this section which, among other things, gives several convenient characterizations of $(n+1)$-cluster subcategories.

Theorem 6.3. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$, and $n \geqslant 1$. Then the following are equivalent.*

(i) *$\mathcal{X}$ is a $(n+1)$-cluster tilting subcategory of $\mathcal{T}$.*
(ii) *$\mathcal{X}$ is contravariantly finite and $\mathcal{X} = \mathcal{X}_n^{\top}$.*
(iii) *$\mathcal{X}$ is covariantly finite and $\mathcal{X} = {}_n^{\top}\mathcal{X}$.*
(iv) *$\mathcal{X}$ is contravariantly finite and both $\mathcal{X}$ and $\mathcal{X}_n^{\top}$ are n-rigid.*
(v) *$\mathcal{X}$ covariantly finite and both $\mathcal{X}$ and ${}_n^{\top}\mathcal{X}$ are n-rigid.*
(vi) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{gl.dim}_{\mathcal{X}}\, \mathcal{T} = n$.*
(vii) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$.*
(viii) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and: $\mathsf{Gh}^{[n+1]}(\mathcal{T}) = 0$.*
(ix) *$\mathcal{X}$ is contravariantly (or covariantly) finite n-rigid and, $\forall A \in \mathcal{T}$: $\Omega^n_{\mathcal{X}}(A) \in \mathcal{X}$.*
(x) *$\mathcal{X}$ is covariantly (or contravariantly) finite n-rigid and, $\forall A \in \mathcal{T}$: $\Sigma^n_{\mathcal{X}}(A) \in \mathcal{X}$.*
(xi) *$\mathcal{X}$ is contravariantly finite n-rigid, any object of $\mathcal{X}_n^{\top}[n+1]$ is injective in mod-$\mathcal{X}$ and the functor $\mathsf{H} : \mathcal{X}_n^{\top}[n+1] \longrightarrow \mathsf{mod}$-$\mathcal{X}$ is full and reflects isomorphisms.*

If $\mathcal{X}$ is a $(n+1)$-cluster subcategory of $\mathcal{T}$, then the abelian category mod-$\mathcal{X}$ has enough projectives and enough injectives, the functors $\mathcal{X}[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X} \longleftarrow \mathcal{X}$ are fully faithful and induce equivalences

$$\mathcal{X}[n+1] \xrightarrow{\approx} \mathsf{Inj\,mod}\text{-}\mathcal{X} \quad \textit{and} \quad \mathcal{X} \xrightarrow{\approx} \mathsf{Proj\,mod}\text{-}\mathcal{X}$$

By Proposition 6.1 and the results of sections 4 and 5, the first ten conditions are equivalent. So to complete the proof, it remains to show that (xi) is equivalent to (i). This requires several steps.

Lemma 6.4. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$. If $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$, $0 \leqslant k \leqslant n$, and $B \in \mathcal{T}$ is such that $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leqslant i \leqslant k-1$, then :*

$$\mathsf{Gh}_{\mathcal{X}}\big(\Omega^1_{\mathcal{X}}(A), B\big) = 0$$

and the map $\mathsf{H}_{A,B} : \mathcal{T}(A,B) \longrightarrow \mathsf{Hom}\big(\mathsf{H}(A), \mathsf{H}(B)\big)$, $f \longmapsto \mathsf{H}(f)$, is surjective.

Proof. Since $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$, $k \leqslant n$, it follows easily that $\Omega^k_{\mathcal{X}}(A) \in \mathcal{X}$ and then by induction we infer that $\Omega^1_{\mathcal{X}}(A) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$. Hence there is a triangle $X \longrightarrow \Omega^1_{\mathcal{X}}(A) \longrightarrow C \longrightarrow X[1]$, where $X \in \mathcal{X}$ and $C \in \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$. Let $\alpha : \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ be an $\mathcal{X}$-ghost map. Then the composition $X \longrightarrow \Omega^1_{\mathcal{X}}(A) \longrightarrow B$ is zero, hence $\Omega^1_{\mathcal{X}}(A) \longrightarrow B$ factorizes through $\Omega^1_{\mathcal{X}}(A) \longrightarrow C$. Since any map from an object from $\mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$ to an object B satisfying $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leqslant i \leqslant k-1$, is clearly zero, it follows that $\mathcal{T}(C, B) = 0$. This implies that the map $\Omega^1_{\mathcal{X}}(A) \longrightarrow B$ is zero and consequently $\mathsf{Gh}_{\mathcal{X}}\big(\Omega^1_{\mathcal{X}}(A), B\big) = 0$. □

Corollary 6.5. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Let $B \in \mathcal{T}$ be such that $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leqslant i \leqslant n-1$. Then for any object $A \in \mathcal{T}$, there exists a short exact sequence*

$$0 \longrightarrow \mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B) \longrightarrow \mathcal{T}(A,B) \longrightarrow \mathsf{Hom}[\mathsf{H}(A), \mathsf{H}(B)] \longrightarrow 0$$

If, in addition, $\mathcal{T}(\mathcal{X}, B[-n]) = 0$, then the map $\mathsf{H}_{A,B}$ is invertible.

Proof. By Lemma 6.4, with $k = n$, we have $\mathsf{Gh}_{\mathcal{X}}\big(\Omega^1_{\mathcal{X}}(A), B\big) = 0$, hence by Proposition 3.6 the map $\mathsf{H}_{A,B}$ is surjective. We show that $\mathsf{Gh}_{\mathcal{X}}(A,B) = \mathsf{Ker}\,\mathsf{H}_{A,B} = \mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B)$. Let $\alpha : A \longrightarrow B$ be such that $\mathsf{H}(\alpha) = 0$, i.e. α is $\mathcal{X}$-ghost. Then α factorizes through $h^1_A : A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$, say via a map $\beta : \Omega^1_{\mathcal{X}}(A)[1] \longrightarrow B$. Since $\Omega^1_{\mathcal{X}}(A)[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$, there are triangles $X_i[i] \xrightarrow{l_i} N_i \xrightarrow{\xi_i} N_{i+1} \longrightarrow X_i[i+1]$, for $1 \leqslant i \leqslant n-1$, where $N_1 = \Omega^1_{\mathcal{X}}(A)[1]$, $X_1 \in \mathcal{X}$ and $M_{i+1} \in \mathcal{X}[i+1] \star \cdots \star \mathcal{X}[n]$; in particular $M_n = X_n[n]$, where $X_n \in \mathcal{X}$. Since $\mathcal{T}(\mathcal{X}, B[-1]) = 0$, we have $l_1 \circ \beta = 0$, and therefore $\beta = \xi_1 \circ \beta_2$ for some map $\beta_2 : N_2 \longrightarrow B$. Using that $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leqslant i \leqslant n-1$, by induction there exists a factorization $\beta = \xi_1 \circ \xi_2 \circ \cdots \xi_{n-1} \circ \beta_n$, where $\beta_n : X_n[n] \longrightarrow B$. Since $M_i \in \mathcal{X}[i] \star \cdots \mathcal{X}[n]$ and since clearly any map from an object of $\mathcal{X}[i]$ to an object in $\mathcal{X}[i+1] \star \cdots \mathcal{X}[n]$ is zero, the map $\xi_i : M_i \longrightarrow M_{i+1}$ is $\mathcal{X}[i]$-ghost. In particular the map $\xi_{n-1} \circ \beta_n : M_{n-1} \longrightarrow B$ is $\mathcal{X}[n-1]$-ghost. Since $\alpha = h^1_A \circ \beta = h^1_A \circ \xi_1 \circ \xi_2 \circ \cdots \xi_{n-1} \circ \beta_n$, it follows that α lies in $\mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B)$, hence $\mathsf{Ker}\,\mathsf{H}_{A,B} \subseteq \mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B)$. Since clearly $\mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B) \subseteq \mathsf{Gh}_{\mathcal{X}}(A,B)$, the assertion follows. If in addition $\mathcal{T}(\mathcal{X}, B[-n]) = 0$, then the map β_n is zero. So $\alpha = 0$ and then $\mathsf{Gh}^{[n]}_{\mathcal{X}}(A,B) = 0$. □

Since any object $B \in \mathcal{X}[n+1]$ satisfies the assumptions of the above Corollary we have the following.

Corollary 6.6. *For any object $A \in \mathcal{T}$ and any object $X \in \mathcal{X}$, we have an isomorphism*

$$\mathsf{H}_{A,X[n+1]} \ : \ \mathcal{T}(A, X[n+1]) \xrightarrow{\cong} \mathsf{Hom}\big(\mathsf{H}(A), \mathsf{H}(X[n+1])\big)$$

In particular the functor $\mathsf{H} : \mathcal{X}[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is fully faithful.

The following result gives the implication (i) $\Rightarrow$ (ix) in Theorem 6.3.

Proposition 6.7. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Then $\mathsf{mod}\text{-}\mathcal{X}$ has enough injectives, and the functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ induces an equivalence*

$$\mathsf{H} \ : \ \mathcal{X}[n+1] \xrightarrow{\approx} \mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X}$$

Proof. Recall that by Lemma 2.4 the functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is almost full, i.e. setting $A^* = \mathsf{Cell}_1(A)$, for any object $A \in \mathcal{T}$, we have a canonical map $\gamma^1_A : A^* \longrightarrow A$ such that $\mathsf{H}(\gamma^1_A)$ is invertible and, for any map $\tilde{\mu} : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$, there there exists a commutative diagram

$$\begin{tikzcd}
\mathsf{H}(A^*) \arrow[r, "\mathsf{H}(\mu)"] \arrow[d, "\mathsf{H}(\gamma^1_A)"', "\cong"] & \mathsf{H}(B^*) \arrow[d, "\cong"', "\mathsf{H}(\gamma^1_B)"] \\
\mathsf{H}(A) \arrow[r, "\tilde{\mu}"] & \mathsf{H}(B)
\end{tikzcd}$$

Let $\tilde{\mu} : \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ be a monomorphism in $\mathsf{mod}\text{-}\mathcal{X}$ and let $\tilde{\alpha} : \mathsf{H}(A) \longrightarrow \mathsf{H}(X[n+1])$ be a map, where $X \in \mathcal{X}$. Clearly the map $\mathsf{H}(\mu)$ is a monomorphism, so if $C \longrightarrow A^* \longrightarrow B^* \longrightarrow C[1]$ is a triangle in $\mathcal{T}$, then the map $C \longrightarrow A^*$ is $\mathcal{X}$-ghost and therefore it factorizes through $\mathcal{X}^\top = \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. By Corollary 6.6, there is a map $\alpha : A \longrightarrow X[n+1]$ such that $\mathsf{H}(\alpha) = \tilde{\alpha}$. Since any map from an object of $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ to an object of $\mathcal{X}[n+1]$ is clearly zero, it follows that the composition $C \longrightarrow A^* \longrightarrow A \longrightarrow X[n+1]$ is zero and therefore $\gamma^1_A \circ \alpha$ factorizes through μ, i.e. $\gamma^1_A \circ \alpha = \mu \circ \rho$ for some map $\rho : B^* \longrightarrow X[n+1]$. Then we have $\mathsf{H}(\gamma^1_A) \circ \mathsf{H}(\alpha) = \mathsf{H}(\mu) \circ \mathsf{H}(\rho)$ and therefore $\mathsf{H}(\gamma^1_A) \circ \mathsf{H}(\alpha) = \mathsf{H}(\gamma^1_A) \circ \tilde{\mu} \circ \mathsf{H}(\gamma^1_B)^{-1} \circ \mathsf{H}(\rho)$, hence $\mathsf{H}(\alpha) = \tilde{\mu} \circ \mathsf{H}(\gamma^1_B)^{-1} \circ \mathsf{H}(\rho)$. This shows that $\mathsf{H}(X[n+1])$ is injective, $\forall X \in \mathcal{X}$.

We show that any object $\mathsf{H}(A)$ of $\mathsf{mod}\text{-}\mathcal{X}$ is a subobject of an object from $\mathsf{H}(\mathcal{X}[n+1])$. There is a map $\omega^{n-1}_{A[-1]} : A \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1]$ and a triangle

$$\Omega^n_{\mathcal{X}}(A[-1]) \ \longrightarrow \ \mathsf{Cell}_{n-1}(A[-1])[1] \ \longrightarrow \ A \ \longrightarrow \ \Omega^n_{\mathcal{X}}(A[-1])[n+1]$$

where by construction $\mathsf{Cell}_{n-1}(A[-1])[1]$ lies in $\mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Since $\Omega^n_{\mathcal{X}}(A[-1]) \in \mathcal{X}$, we have $\Omega^n_{\mathcal{X}}(A[-1])[n+1] \in \mathcal{X}[n+1]$. Hence $\mathsf{H}(\Omega^n_{\mathcal{X}}(A[-1])[n+1])$ is injective in $\mathsf{mod}\text{-}\mathcal{X}$, and the map $\mathsf{H}(\omega^{n-1}_{A[-1]}) : \mathsf{H}(A) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(A[-1])[n+1])$ is a monomorphism, since $\mathsf{H}(\mathcal{X}[1] \star \cdots \star \mathcal{X}[n]) = 0$. hence $\mathsf{mod}\text{-}\mathcal{X}$ has enough injectives.

Now let $\mathsf{H}(A)$ be an injective object of $\mathsf{mod}\text{-}\mathcal{X}$. By the above there exists a split monomorphism $\mathsf{H}(\mu) : \mathsf{H}(A) \longrightarrow \mathsf{H}(X[n+1])$, where $X \in \mathcal{X}$. Hence there exists a map $\tilde{\alpha} : \mathsf{H}(X[n+1]) \longrightarrow \mathsf{H}(A)$ such that $\mathsf{H}(\mu) \circ \tilde{\alpha} = 1_{\mathsf{H}(A)}$. Now the map $\tilde{\alpha} \circ \mathsf{H}(\mu)$ is an idempotent endomorphism of $\mathsf{H}(X[n+1])$ and therefore since the functor $\mathsf{H} : \mathcal{X}[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is fully faithful, there exists an idempotent endomorphism $e : X[n+1] \longrightarrow X[n+1]$ such that $\mathsf{H}(e) = \tilde{\alpha} \circ \mathsf{H}(\mu)$. Since idempotents split in $\mathcal{T}$, there exist maps $\kappa : X[n+1] \longrightarrow D$ and $\lambda : D \longrightarrow X[n+1]$ such that $e = \kappa \circ \lambda$ and $\lambda \circ \kappa = 1_{X[n+1]}$. Clearly D is of the form $X'[n+1]$, for some object $X' \in \mathcal{X}$, as a direct summand of $X[n+1]$. We claim that the map $\phi := \mathsf{H}(\mu) \circ \mathsf{H}(\kappa) \ : \ \mathsf{H}(A) \longrightarrow \mathsf{H}(X'[n+1])$ is an isomorphism with inverse the map $\psi := \mathsf{H}(\lambda) \circ \tilde{\alpha}$. Indeed we have:

$$\phi \circ \psi = \mathsf{H}(\mu) \circ \mathsf{H}(\kappa) \circ \mathsf{H}(\lambda) \circ \tilde{\alpha} = \mathsf{H}(\mu) \circ \mathsf{H}(\kappa \circ \lambda) \circ \tilde{\alpha} = \mathsf{H}(\mu) \circ \mathsf{H}(e) \circ \tilde{\alpha} = \mathsf{H}(\mu) \circ \tilde{\alpha} \circ \mathsf{H}(\mu) \circ \tilde{\alpha} = 1_{\mathsf{H}(A)}$$

$$\psi \circ \phi = \mathsf{H}(\lambda) \circ \tilde{\alpha} \circ \mathsf{H}(\mu) \circ \mathsf{H}(\kappa) = \mathsf{H}(\lambda) \circ \mathsf{H}(e) \circ \mathsf{H}(\kappa) = \mathsf{H}(\lambda) \circ \mathsf{H}(\kappa) \circ \mathsf{H}(\lambda) \circ \mathsf{H}(\kappa) = 1_{\mathsf{H}(X'[n+1])}$$

Hence the functor $\mathsf{H} : \mathcal{X}[n+1] \longrightarrow \mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X}$ is surjective on objects and therefore an equivalence. ☐

Finally the next result shows the implication (ix) $\Rightarrow$ (i) and completes the proof of Theorem 6.3.

Proposition 6.8. *Let $\mathcal{X}$ be a contravariantly finite n-rigid subcategory of $\mathcal{T}$. If the functor $\mathsf{H} : \mathcal{X}^\top_n[n+1] \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ has image in $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X}$, is full and reflects isomorphisms, then $\mathcal{X}$ is $(n+1)$-cluster tilting.*

Proof. It suffices to show that $\mathcal{X}^\top_n \subseteq \mathcal{X}$. Let $A \in \mathcal{X}^\top_n$ and consider the triangle $\Omega^1_{\mathcal{X}}(A) \xrightarrow{g^0_A} X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. Applying H and using that $\mathcal{X}$ is n-rigid and $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leqslant i \leqslant n$, it follows that $\mathcal{T}(\mathcal{X}, \Omega^1_{\mathcal{X}}(A)[i]) = 0$, $1 \leqslant i \leqslant n$, so $\Omega^1_{\mathcal{X}}(A) \in \mathcal{X}^\top_n$, and we have a monomorphism $\mathsf{H}(g^0_A[n+1]) : \mathsf{H}(\Omega^1_{\mathcal{X}}(A)[n+1]) \longrightarrow \mathsf{H}(X^0_A[n+1])$. On the other hand since $\mathcal{X} \subseteq \mathcal{X}^\top_n$ it follows that $X^0_A[n+1] \in \mathcal{X}^\top_n[n+1]$. Since the objects

$\mathsf{H}(\Omega^1_{\mathcal{X}}(A)[n+1])$ and $\mathsf{H}(X^0_A[n+1])$ are injective in $\mathsf{mod}\text{-}\mathcal{X}$, the above monomorphism splits. Since by hypothesis $\mathsf{H}|_{\mathcal{X}^{\top}_n[n+1]}$ is full and reflects isomorphisms, this implies that the map $g^0_A[n+1] : \Omega^1_{\mathcal{X}}(A)[n+1] \longrightarrow X^0_A[n+1]$, or equivalently the map $\Omega^1_{\mathcal{X}}(A)[1] \longrightarrow X^0_A[1]$, is split monic. Then the map $A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$ is zero and therefore A lies in $\mathcal{X}$ as a direct summand of $X^0_A \in \mathcal{X}$. Hence $\mathcal{X}^{\top}_n = \mathcal{X}$. $\square$

From now on let $\mathcal{X}$ be a $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$.

Corollary 6.9. *The abelian category* $\mathsf{mod}\text{-}\mathcal{X}$ *is Frobenius if and only if* $\mathcal{X} = \mathcal{X}[n+1]$.

Corollary 6.10. *There exists a torsion triple:*

$$\big(\mathcal{X}[k-n] \star \cdots \star \mathcal{X}[k-2] \star \mathcal{X}[k-1],\ \mathcal{X}[k],\ \mathcal{X}[k+1] \star \mathcal{X}[k+2] \star \cdots \star \mathcal{X}[k+n]\big), \quad \forall k \in \mathbb{Z}$$

Proof. Since $\mathcal{X}^{\top}_n = \mathcal{X}$, by Proposition 4.2 there is a torsion pair $\big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1], \mathcal{X}[n]\big)$ in $\mathcal{T}$. Clearly then $\big(\mathcal{X}[k-n] \star \cdots \star \mathcal{X}[k-2] \star \mathcal{X}[k-1],\ \mathcal{X}[k]\big)$ is a torsion pair in $\mathcal{T}$, $\forall k \in \mathbb{Z}$. The proof that $\big(\mathcal{X}[k],\ \mathcal{X}[k+1] \star \mathcal{X}[k+2] \star \cdots \star \mathcal{X}[k+n]\big)$ is a torsion pair in $\mathcal{T}$ is dual, using cellular cotowers, see 2.2. $\square$

We denote by $\mathsf{K}_0(\mathcal{X}, \oplus)$ the split Grothendieck group of the exact category $\mathcal{X}$ endowed with the split exact structure and by $\mathsf{K}_0(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$. If $n = 1$, so that $\mathcal{X}$ is a 2-cluster tilting subcategory, and if $\mathcal{T}$ is algebraic, then Palu [25] proved that $\mathsf{K}_0(\mathcal{T})$ is a quotient of $\mathsf{K}_0(\mathcal{X}, \oplus)$ by a certain subgroup. In our case we have the following for $n \geqslant 2$ and for arbitrary $\mathcal{T}$.

Corollary 6.11. *The inclusion* $\mathcal{X} \longrightarrow \mathcal{T}$ *induces an epimorphism:* $\mathsf{K}_0(\mathcal{X}, \oplus) \longrightarrow \mathsf{K}_0(\mathcal{T}) \longrightarrow 0$.

Proof. Clearly the inclusion $i : \mathcal{X} \longrightarrow \mathcal{T}$ induces an homomorphism $\mathsf{K}_0(i) : \mathsf{K}_0(\mathcal{X}, \oplus) \longrightarrow \mathsf{K}_0(\mathcal{T})$, by $\mathsf{K}_0(i)[X] = [X]$. Let A be in $\mathcal{T}$ and consider the triangle $\Omega^1_{\mathcal{X}}(A) \longrightarrow X^0_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}(A)[1]$. Then in $\mathsf{K}(\mathcal{T})$ we have a relation $[A] = [X^0_A] - [\Omega^1_{\mathcal{X}}(A)]$. Similarly the triangle $\Omega^2_{\mathcal{X}}(A) \longrightarrow X^1_A \longrightarrow \Omega^1_{\mathcal{X}}(A) \longrightarrow \Omega^2_{\mathcal{X}}(A)[1]$ gives the relation $[\Omega^1_{\mathcal{X}}(A)] = [X^1_A] - [\Omega^2_{\mathcal{X}}(A)]$ and therefore we have $[A] = [X^0_A] - [X^1_A] + [\Omega^2_{\mathcal{X}}(A)]$. Continuing in this way we have a relation in $\mathsf{K}_0(\mathcal{T})$: $[A] = \sum_{i=0}^{n-1}(-1)^i[X^i_A] + (-1)^n[\Omega^n_{\mathcal{X}}(A)]$. By Theorem 5.3 we have $\Omega^n_{\mathcal{X}}(A) := X^n_A \in \mathcal{X}$, hence $[A] = \sum_{i=0}^{n}(-1)^i[X^i_A]$. Hence $[A] = \sum_{i=0}^{n}(-1)^i\mathsf{K}_0(i)\big([X^i_A]\big) = \mathsf{K}_0(i)\big(\sum_{i=0}^{n}(-1)^i[X^i_A]\big)$, i.e. for any object A in $\mathcal{T}$, the generator $[A]$ lies in the image of $\mathsf{K}_0(i)$, This clearly implies that $\mathsf{K}_0(i)$ is surjective. $\square$

7. Certain Cluster Tilting Subcategories are Gorenstein

Our aim in this section is to show that a special class of $(n+1)$-cluster tilting subcategories, called $(n-k)$-strong $(n+1)$-cluster tilting subcategories, of an arbitrary triangulated category, where $n \geqslant 2k-1$, enjoys the property that the associated cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein.

Main examples include all 2-cluster tilting subcategories and all $(n+1)$-cluster categories of associated to a finite-dimensional hereditary algebra over a field.

7.1. **Gorenstein Categories.** Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. We recall from [12] the following invariants attached to $\mathscr{A}$:

$$\mathsf{silp}\mathscr{A} = \sup\{\mathsf{id}\, P \mid P \in \mathsf{Proj}\,\mathscr{A}\}, \qquad \mathsf{spli}\mathscr{A} = \sup\{\mathsf{pd}\, I \mid I \in \mathsf{Inj}\,\mathscr{A}\}$$

$$\mathsf{G\text{-}dim}\,\mathscr{A} := \max\big\{\mathsf{silp}\mathscr{A}, \mathsf{spli}\mathscr{A}\big\}$$

We call $\mathsf{G\text{-}dim}\,\mathscr{A}$ the *Gorenstein dimension* of $\mathscr{A}$ and then $\mathscr{A}$ is called *Gorenstein* if $\mathsf{G\text{-}dim}\,\mathscr{A} < \infty$. If $\mathsf{G\text{-}dim}\,\mathscr{A} \leqslant n < \infty$, then we say that $\mathscr{A}$ is *n-Gorenstein*.

Lemma 7.1. *Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. Assume that* $\mathsf{spli}\,\mathscr{A} < \infty$ *and* $\mathsf{silp}\,\mathscr{A} < \infty$. *Then $\mathscr{A}$ is Gorenstein of dimension* $\mathsf{G\text{-}dim}\,\mathscr{A} = \mathsf{spli}\,\mathscr{A} = \mathsf{silp}\,\mathscr{A}$.

Proof. Let $\mathsf{spli}\,\mathscr{A} = n < \infty$ and $\mathsf{silp}\,\mathscr{A} = m < \infty$. If $m = 0$, then any projective object of $\mathscr{A}$ is injective, and since any injective has finite projective dimension, bounded by n, it follows that any injective object is projective, i.e. $n = 0$. Dually if $n = 0$ we have $m = 0$. In both cases $\mathscr{A}$ is Frobenius, i.e. $\mathscr{A}$ is Gorenstein of dimension $\mathsf{G\text{-}dim}\,\mathscr{A} = 0$. Now let $n > m > 0$, so $n \geqslant 2$. Since $\mathsf{spli}\,\mathscr{A} = n$ there exists an injective object I such that $\Omega^{n-1}I$ is not projective, so $\mathsf{Ext}^1_{\mathscr{A}}(\Omega^{n-1}I, -) \neq 0$. Hence there exists an object $A \in \mathscr{A}$ such that $\mathsf{Ext}^n_{\mathscr{A}}(I, A) = \mathsf{Ext}^1_{\mathscr{A}}(\Omega^{n-1}I, A) \neq 0$. Let $0 \longrightarrow \Omega(A) \longrightarrow P \longrightarrow A \longrightarrow 0$ be exact where P is projective. Evaluating the exact sequence of functors $\cdots \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(-, P) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(-, A) \longrightarrow \mathsf{Ext}^2_{\mathscr{A}}(-, \Omega A) \longrightarrow \cdots$ at $\Omega^{n-1}I$ and using that $\mathsf{Ext}^2_{\mathscr{A}}(\Omega^{n-1}I, \Omega A) \cong \mathsf{Ext}^{n+1}_{\mathscr{A}}(I, \Omega A) = 0$ since $\mathsf{pd}\, I \leqslant n$, we get an epimorphism $\mathsf{Ext}^1_{\mathscr{A}}(\Omega^{n-1}I, P) \longrightarrow \mathsf{Ext}^1_{\mathscr{A}}(\Omega^{n-1}I, A) \longrightarrow 0$. Hence $\mathsf{Ext}^n_{\mathscr{A}}(I, P) = \mathsf{Ext}^1_{\mathscr{A}}(\Omega^{n-1}I, P) \neq 0$. This shows that $m \geqslant \mathsf{id}\, P > n$ and this is not the case. Dually if $m > n$, then we arrive at a contradiction. This shows that $n = m$ and therefore $\mathscr{A}$ is Gorenstein of Gorenstein dimension $\mathsf{G\text{-}dim}\,\mathscr{A} = \mathsf{spli}\,\mathscr{A} = \mathsf{silp}\,\mathscr{A}$. $\square$

7.2. **Strong Cluster Tilting Subcategories.** Let as before $\mathcal{T}$ be a triangulated category with split idempotents and $\mathcal{X}$ a full subcategory of $\mathcal{T}$ closed under direct summands and isomorphisms.

Then we have a chain of extension closed full subcategories of $\mathcal{T}$:

$$\cdots \subseteq \mathcal{X}^{\top}_{n-1}[n] \subseteq \mathcal{X}^{\top}_{n-2}[n-1] \subseteq \mathcal{X}^{\top}_{2}[3] \subseteq \mathcal{X}^{\top}_{1}[2] \subseteq \mathcal{X}^{\top}_{0}[1] = \mathcal{X}[2] \star \mathcal{X}[3] \star \cdots \star \mathcal{X}[n+1] \qquad (*)$$

where, as easily seen: $\mathcal{X}^{\top}_{t}[t+1] = \big\{A \in \mathcal{T} \mid \mathcal{T}(\mathcal{X}, A[-k]) = 0, \ 1 \leqslant k \leqslant t\big\}$.

Clearly if $\mathcal{X}$ is $(n+1)$-cluster tilting, then: $\mathcal{X}^{\top}_{n+1}[n+2] = 0$ and $\mathcal{X}[n+1] = \mathcal{X}^{\top}_{n}[n+1]$.

Definition 7.2. A full subcategory $\mathcal{X}$ of $\mathcal{T}$ is called t**-strong**, where $t \geqslant 1$, if:

$$\mathcal{X} \subseteq \mathcal{X}^{\top}_{t}[t+1], \quad \text{i.e.} \quad \mathcal{T}(\mathcal{X}, \mathcal{X}[-1]) = \cdots = \mathcal{T}(\mathcal{X}, \mathcal{X}[-t]) = 0$$

The following gives a convenient characterization of when a cluster tilting subcategory is t-strong.

Proposition 7.3. *If $\mathcal{X}$ is $(n+1)$-cluster tilting, then for $1 \leqslant t \leqslant n$, the following are equivalent:*

(i) $\mathcal{X}$ *is t-strong.*
(ii) $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[n-t]$.

Proof. (i) $\Rightarrow$ (ii) We know that $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ and therefore $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$. Hence for any object $X \in \mathcal{X}$, there exists a triangle $A \longrightarrow X[n+1] \longrightarrow B \longrightarrow A[1]$, where $A \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t]$ and $B \in \mathcal{X}[n-t+1] \star \cdots \star \mathcal{X}[n]$. Now the hypothesis (ii) implies that any map from an object from $\mathcal{X}$ to an object from $\mathcal{X}[-t] \star \cdots \star \mathcal{X}[-1]$ is zero and this trivially implies that any map from an object from $\mathcal{X}[n+1]$ to an object from $\mathcal{X}[n-t+1] \star \cdots \star \mathcal{X}[n]$ is zero. It follows that the map $X[n+1] \longrightarrow B$ is zero, hence $X[n+1]$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-t]$ as a direct summand of A.

(ii) $\Rightarrow$ (i) The hypothesis implies that $\mathcal{X} \subseteq \mathcal{X}[-n-1] \star \mathcal{X}[-n] \star \cdots \star \mathcal{X}[-t-1]$. Hence for any object $X \in \mathcal{X}$, there exists a triangle $A \longrightarrow X \longrightarrow B \longrightarrow A[1]$, where $A \in \mathcal{X}[-n-1] \star \cdots \star \mathcal{X}[-t-1]$. Let $1 \leqslant k \leqslant t$ and consider any map $X \longrightarrow X'[-k]$, where $X' \in \mathcal{X}$. The composition $A \longrightarrow X \longrightarrow X'[-k]$ is zero since it lies in $\mathcal{T}(\mathcal{X}, \mathcal{X}[n+1-k]$. Hence the map $X \longrightarrow X'[-k]$ factorizes through the map $X \longrightarrow B$. However using that $1 \leqslant k \leqslant t$, it follows easily that any map from an object from $\mathcal{X}[-n] \star \cdots \star \mathcal{X}[-t-1]$ to an object from $\mathcal{X}[-k]$ is zero. Hence the map $\mathcal{X} \longrightarrow X'[-k])$ is zero, i.e. $\mathcal{T}(\mathcal{X}, \mathcal{X}[-k]) = 0, \ \ 1 \leqslant k \leqslant t$. $\square$

Corollary 7.4. *Let $\mathcal{X}$ be an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $0 \leqslant k \leqslant n-1$.*

(i) $\mathcal{X}[n+t+1] \subseteq \mathcal{X}[t] \star \mathcal{X}[t+1] \star \cdots \star \mathcal{X}[t+k], \quad \forall t \in \mathbb{Z}$
(ii) $\mathcal{T}(\mathcal{X}, \mathcal{X}[n+t+1]) = 0,$ *for* $1 \leqslant t \leqslant n-k$.

Proof. By Proposition 7.3 we have $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$ and (i) follows. Now $\mathcal{T}(\mathcal{X}, \mathcal{X}[n+t+1])$ is contained in $\mathcal{X}[t] \star \cdots \star \mathcal{X}[n+k]$ and clearly $\mathcal{X}[t] \star \cdots \star \mathcal{X}[n+k] \subseteq \mathcal{X}[1] \star \cdots \star \mathcal{X}[n] = \mathcal{X}^{\top}$ for $1 \leqslant t \leqslant n-k$. $\square$

7.3. **Gorensteinness of Strong Cluster Tilting Subcategories.** Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Let $0 \leqslant k \leqslant n-1$ we say that $\mathcal{X}$ is *strictly $(n-k)$-strong* if $\mathcal{X}$ is $(n-k)$-strong but $\mathcal{X}$ is not $(n-k+1)$-strong. Our main aim in this section is to prove the following result. Note that (i) is due to Keller-Reiten [21] and the case $k=1$ in (iii) was observed independently by Iyama-Oppermann [17].

Theorem 7.5. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$.*

(i) *If $n=1$, then* $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} \leqslant 1$.
(ii) $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = 0$ *if and only if* $\mathcal{X}$ *is n-strong.*
(iii) *Assume that $n \geqslant 2$ and $\mathcal{X}$ is $(n-k)$-strong, where $0 \leqslant k \leqslant n-1$. Then:*

$$0 \leqslant k \leqslant \frac{n+1}{2} \qquad \Longrightarrow \qquad \mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} \leqslant k$$

In particular:

(a) *If n is odd and $\mathcal{X}$ is $(\frac{n-1}{2})$-strong, then:* $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} \leqslant \frac{n+1}{2}$.
(b) *If n is even and $\mathcal{X}$ is $(\frac{n+1}{2})$-strong, then:* $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} \leqslant \frac{n-1}{2}$.

Moreover if $\mathcal{X}$ is strictly $(n-k)$-strong, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = k$.

We split the proof of Theorem 7.5 into three steps as Propositions 7.7, 7.8 and 7.9.

7.3.1. *Finiteness of* $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X}$. We first investigate when $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} < \infty$. We begin with the reformulation of Theorem 5.4.

Proposition 7.6. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{T}$ and $A \in \mathcal{T}$.*

(i) *If $\mathcal{X}$ is 1-rigid and $A \in \big(\mathcal{X} \star \mathcal{X}[1]\big) \cap \mathcal{X}_1^{\top}[2]$, then:* $\mathsf{pd}\,\mathsf{H}(A) \leqslant 1$.

(ii) *Let $t \geqslant 2$ and assume that $\mathcal{X}$ is $(t-1)$-strong and t-rigid. Then:*

$$A \in \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\big) \cap \mathcal{X}_t^{\top}[t+1] \quad \Longrightarrow \quad \mathsf{pd}\,\mathsf{H}(A) \leqslant t.$$

The next result proves half of Theorem 7.5.

Proposition 7.7. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Let $0 \leqslant k \leqslant n-1$ and assume that $\mathcal{X}$ is $(n-k)$-strong. Then $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$ provided that $n \geqslant 2k-1$.*

Proof. If $\mathcal{X}$ is n-strong, then clearly $\mathcal{X}[n+1] = \mathcal{X}_n^{\top} = \mathcal{X}$ and therefore any injective object of $\mathsf{mod}\text{-}\mathcal{X}$ is projective. Hence $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} = 0$. Assume that $\mathcal{X}$ is $(n-1)$-strong. If $n = 1$, then the condition 0-strong is vacuous, but $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1]$, since $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1]$. Hence for any $X \in \mathcal{X}$, there exists a triangle $X^1 \longrightarrow X^0 \longrightarrow X[2] \longrightarrow X^1[1]$, where $X^1, X^1 \in \mathcal{X}$, This clearly implies that $\mathsf{pd}\,\mathsf{H}(X[2]) \leqslant 1$, i.e. $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 1$. If $n \geqslant 2$, then by Proposition 1.3 we have $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1]$ and clearly $\mathcal{X}[n+1] \subseteq \mathcal{X}_1^{\top}[2]$. Then by Proposition 1.2 we have $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant 1$, $\forall X \in \mathcal{X}$. Hence $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 1$. If $\mathcal{X}$ is $(n-2)$-strong and $n \geqslant 3$, then by Proposition 1.3 we have $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2]$, and clearly $\mathcal{X} \subseteq \mathcal{X}_2^{\top}[3]$, since $n \geqslant 3$. Then by Proposition 7.6 we have $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant 2$, $\forall X \in \mathcal{X}$. Hence $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 2$.

Continuing in this way assume that $\mathcal{X}$ is $(n-k)$-strong, where $2 \leqslant k \leqslant n-1$. By Proposition 7.3 then we have $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$. Since clearly $\mathcal{X}[n+1] \subseteq \mathcal{X}_k^{\top}[k+1]$, we have $\mathcal{X}[n+1] \subseteq \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\big) \cap \mathcal{X}_k^{\top}[k+1]$. Our assumption $n \geqslant 2k-1$ gives $n-k \geqslant k-1$, so the chain of subcategories $(*)$ shows that $\mathcal{X}_{n-k}^{\top}[n-k+1] \subseteq \mathcal{X}_{k-1}^{\top}[k]$. Since $\mathcal{X}$ is $(n-k)$-strong, it follows that $\mathcal{X} \subseteq \mathcal{X}_{n-k}^{\top}[n-k+1]$ and therefore $\mathcal{X} \subseteq \mathcal{X}_{k-1}^{\top}[k]$, i.e. $\mathcal{X}$ is $(k-1)$-strong. Then Proposition 7.6 gives us that $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant k$, $\forall X \in \mathcal{X}$. We infer that $\mathsf{spli}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$. □

7.3.2. *Finiteness of* $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X}$. Now we turn our attention to the investigation of when $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X}$ is finite. The following result combined with Proposition 7.7 proves the other half of Theorem 7.5.

Proposition 7.8. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Let $0 \leqslant k \leqslant n-1$ and assume that $\mathcal{X}$ is $(n-k)$-strong. Then $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$ provided that $n \geqslant 2k-1$.*

Proof. Let $X \in \mathcal{X}$ be an arbitrary object. We shall show that $\mathsf{id}\,\mathsf{H}(X) \leqslant k$. For the convenience of the reader and to make the proof more transparent, we first treat the cases $0 \leqslant k \leqslant 2$.

Case $k = 0$. Let first $k = 0$, i.e. $\mathcal{X}$ is n-strong. Then clearly we have $\mathcal{X}[n+1] = \mathcal{X}_n^{\top} = \mathcal{X}$, so by Corollary 6.9, $\mathsf{mod}\text{-}\mathcal{X}$ is Frobenius and then $\mathsf{G}\text{-}\mathsf{dim}\,\mathsf{mod}\text{-}\mathcal{X} = 0$.

Case $k = 1$. Now assume that $n \geqslant 1$ and $k = 1$, so $\mathcal{X}$ is $(n-1)$-strong. First we treat the case $n = 1$. Then the condition 0-strong is vacuous. For any $X \in \mathcal{X}$ we have a triangle

$$\Omega^1_{\mathcal{X}}(X[-1]) \longrightarrow \mathsf{Cell}_0(X[-1]) \longrightarrow X[-1] \longrightarrow \Omega^1_{\mathcal{X}}(X[-1])[1]$$

where $\Omega^1_{\mathcal{X}}(X[-1]) := X_0^* \in \mathcal{X}$ and $\mathsf{Cell}_0(X[-1]) := X_1^* \in \mathcal{X}$. Then we have a triangle

$$X \;\longrightarrow\; X_0^*[2] \;\longrightarrow\; X_1^*[2] \;\longrightarrow\; X[1]$$

which induces a short exact sequence

$$0 \;\longrightarrow\; \mathsf{H}(X) \;\longrightarrow\; \mathsf{H}(X_0^*[2]) \;\longrightarrow\; \mathsf{H}(X_1^*[2]) \;\longrightarrow\; 0$$

and the objects $\mathsf{H}(X_0^*[2])$, $\mathsf{H}(X_1^*[2])$ are injective in $\mathsf{mod}\text{-}\mathcal{X}$. Hence $\mathsf{id}\,\mathsf{H}(X) \leqslant 1$ and therefore $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 1$.

Now assume that $n \geqslant 2$ and therefore we have: $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant n-1$. Consider the triangle arising from the tower $\mathsf{Cell}_{X[-1]}$:

$$\Omega^n_{\mathcal{X}}(X[-1])[n-1] \longrightarrow \mathsf{Cell}_{n-1}(X[-1]) \longrightarrow X[-1] \longrightarrow \Omega^n_{\mathcal{X}}(X[-1])[n]$$

Setting $\Omega^n_{\mathcal{X}}(X[-1]) := X_0^*$ and $C_1 = \mathsf{Cell}_{n-1}(X[-1]))$, we have a triangle

$$X_0^*[n] \longrightarrow C_1[1] \longrightarrow X \longrightarrow X^*[n+1] \tag{7.1}$$

and we know that $X_0^* \in \mathcal{X}$ and $C \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$. Applying H to (7.1) and using that $\mathsf{H}(C_1[1]) = 0$ and the fact that $\mathcal{X}$ is $(n-1)$-strong, so $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant n-1$, we have an exact sequence

$$0 \;\longrightarrow\; \mathsf{H}(X) \;\longrightarrow\; \mathsf{H}(X_0^*[n+1]) \longrightarrow \mathsf{H}(C_1[2]) \;\longrightarrow\; 0 \tag{7.2}$$

and $\mathsf{H}(C_1[-i]) = 0$, for $-1 \leqslant i \leqslant n-2$. It follows that for the object $C_1[-n+1]$ we have $\mathcal{T}(\mathcal{X}, C_1[-n+1][1]) = \mathcal{T}(\mathcal{X}, C_1[-n+1][2]) = \cdots = \mathcal{T}(\mathcal{X}, C_1[-n+1][n]) = 0$, i.e.

$$C_1 \in \mathcal{X}_n^{\top}[n-1] = \mathcal{X}[n-1]$$

It follows that $C_1[2] \in \mathcal{X}[n+1]$. We infer that the object $\mathsf{H}(C_1[2])$ is injective in $\mathsf{mod}\text{-}\mathcal{X}$ and therefore the short exact sequence (7.2) is an injective coresolution of $\mathsf{H}(X)$, i.e. $\mathsf{id}\,\mathsf{H}(X) \leqslant 1$. Hence $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 1$.

Case $k = 2$**.** Next assume that $n \geqslant 3$ and $k = 2$, so $\mathcal{X}$ is $(n-2)$-strong. Equivalently $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant n-2$. Then as above we have a triangle

$$X \longrightarrow X_0^*[n+1] \longrightarrow C_1[2] \longrightarrow X[1] \tag{7.3}$$

where $X_0^* \in \mathcal{X}$ and $C_1 \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$, such that the following sequence is exact:

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(X_0^*[n+1]) \longrightarrow \mathsf{H}(C_1[2]) \longrightarrow 0$$

Applying H to (7.3) and using that $C_1[1] \in \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ and the fact that $\mathcal{X}$ is $(n-2)$-strong, the induced long exact sequence shows that $\mathcal{T}(\mathcal{X}, C_1[1]) = \mathcal{T}(\mathcal{X}, C_1) = \mathcal{T}(\mathcal{X}, C_1[-1]) = \cdots \mathcal{T}(\mathcal{X}, C_1[-n+3]) = 0$, i.e.

$$C_1 \in \mathcal{X}_{n-1}^{\top}[n-2]$$

Now consider the object $C_1[2]$ and let

$$\Omega^n_{\mathcal{X}}(C_1[1])[n-1] \longrightarrow \mathsf{Cell}_{n-1}(C_1[1]) \longrightarrow C_1[1] \longrightarrow \Omega^n_{\mathcal{X}}(C_1[1])[n]$$

be the triangle arising from the tower $\mathsf{Cell}_{C_1[2][-1]} = \mathsf{Cell}_{C_1[1]}$. Setting $\Omega^n_{\mathcal{X}}(C_1[1]) := X_1^*$ and $C_2 := \mathsf{Cell}_{n-1}(C_1[1])$, we have a triangle

$$C_1[2] \longrightarrow X_1^*[n+1] \longrightarrow C_2[2] \longrightarrow C_1[3] \tag{7.4}$$

where $X_1^* \in \mathcal{X}$ and $C_1, C_2 \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$. We claim that $\mathsf{H}(C_1[3]) = 0$ and $C_2[2] \in \mathcal{X}[n+1]$. First note that since $\mathcal{X}$ is $(n-2)$-strong, by Proposition 7.3 it follows that $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-(n-2)] = \mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2]$. Hence $\mathcal{X}[n+2] \subseteq \mathcal{X}[1] \star \mathcal{X}[2] \star \mathcal{X}[3] \subseteq \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n] = \mathcal{X}^{\top}$. Since $n \geqslant 3$, it follows that $\mathsf{H}(\mathcal{X}[n+3]) = 0$ and therefore applying H to the triangle (1.2) we infer that $\mathsf{H}(C_1[3]) = 0$. On the other hand applying H to the triangle (7.4) and using that $\mathcal{X}$ is $(n-2)$-strong and $C_1 \in \mathcal{X}_{n-1}^{\top}[n-2]$, we see that $\mathcal{T}(\mathcal{X}, C_2[-i]) = 0$, $-1 \leqslant i \leqslant n-2$. This means that $C_2[-n+1] \in \mathcal{X}_n^{\top} = \mathcal{X}$ and therefore

$$C_2 \in \mathcal{X}_n^{\top}[n-1] = \mathcal{X}[n-1]$$

It follows that $C_2[2] \in \mathcal{X}[n+1]$. Therefore the triangle (7.4) induces a short exact sequence

$$0 \longrightarrow \mathsf{H}(C_1[2]) \longrightarrow \mathsf{H}(X_1^*[n+1]) \longrightarrow \mathsf{H}(C_2[2]) \longrightarrow 0$$

where the object $\mathsf{H}(C_2[2])$ is injective in $\mathsf{mod}\text{-}\mathcal{X}$. We infer that the exact sequence

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(X_0^*[n+1]) \longrightarrow \mathsf{H}(X_1^*[n+1]) \longrightarrow \mathsf{H}(C_2[2]) \longrightarrow 0$$

is an injective resolution of $\mathsf{H}(X)$ and therefore $\mathsf{id}\,\mathsf{H}(X) \leqslant 2$. Hence $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 2$.

Case $k \leqslant n-1$**.** Now we treat the general case, so assume $0 \leqslant k \leqslant n-1$ and $n \geqslant 2k-1$. Working as above we may construct triangles

$$X \longrightarrow X_0^*[n+1] \longrightarrow C_1[2] \longrightarrow X[1] \tag{7.5}$$

$$C_1[2] \longrightarrow X_1^*[n+1] \longrightarrow C_2[2] \longrightarrow C_1[3] \tag{7.6}$$

$$\vdots$$

$$C_{k-1}[2] \longrightarrow X_{k-1}^*[n+1] \longrightarrow C_k[2] \longrightarrow C_{k-1}[3] \tag{7.7}$$

where $X_i^* \in \mathcal{X}$, for $0 \leqslant i \leqslant k-1$, and $C_i \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$, for $0 \leqslant i \leqslant k$. It follows that $C_i[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n] = \mathcal{X}^{\top}$, so the above triangles induce exact sequences:

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(X_0^*[n+1]) \longrightarrow \mathsf{H}(C_1[2]) \longrightarrow 0 \tag{7.8}$$

$$0 \longrightarrow \mathsf{H}(C_1[2]) \longrightarrow \mathsf{H}(X_1^*[n+1]) \longrightarrow \mathsf{H}(C_2[2]) \longrightarrow \mathsf{H}(C_1[3]) \longrightarrow \cdots \tag{7.9}$$

$$\vdots$$

$$0 \longrightarrow \mathsf{H}(C_{k-1}[2]) \longrightarrow \mathsf{H}(X_{k-1}^*[n+1]) \longrightarrow \mathsf{H}(C_k[2]) \longrightarrow \mathsf{H}(C_{k-1}[3]) \longrightarrow \cdots \tag{7.10}$$

Clearly then to show that $\mathsf{id}\,\mathsf{H}(X) \leqslant k$, it suffices to show that:

$$\mathsf{H}(C_i[3]) = 0, \quad 1 \leqslant i \leqslant k-1 \quad \text{and} \quad C_k[2] \in \mathcal{X}[n+1] \tag{†}$$

Since $\mathcal{X}$ is $(n-k)$-strong, by Proposition 7.3 we have $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]$. It follows that

$$\mathcal{X}[n+i] \subseteq \mathcal{X}[i-1] \star \mathcal{X}[i] \star \cdots \star \mathcal{X}[k+i-1]$$

and therefore

$$\mathcal{X}[n+i] \subseteq \mathcal{X}^{\top}, \quad k \leqslant n-i+1, \quad 2 \leqslant i \leqslant k-1 \tag{7.11}$$

Applying H to the triangle (7.5) and Using (7.11) and the fact that $C_1[1] \in \mathcal{X}^{\top}$, we see directly that

$$\mathsf{H}(C_1[-n+k+1]) = \cdots = \mathsf{H}(C_1) = \mathsf{H}(C_1[1]) = \mathsf{H}(C_1[3]) = \cdots = \mathsf{H}(C_1[k]) = 0 \tag{7.12}$$

Then using (7.12) it follows that (7.9) becomes a short exact sequence

$$0 \longrightarrow \mathsf{H}(C_1[2]) \longrightarrow \mathsf{H}(X_1^*[n+1]) \longrightarrow \mathsf{H}(C_2[2]) \longrightarrow 0 \tag{7.13}$$

and the long exact sequence induced after applying H to the triangle (7.6), gives us

$$\mathsf{H}(C_2[-n+k]) = \cdots = \mathsf{H}(C_2) = \mathsf{H}(C_2[1]) = \mathsf{H}(C_2[3]) = \cdots = \mathsf{H}(C_2[k-1]) = 0 \tag{7.14}$$

Continuing inductively in this way we see $\mathsf{H}(C_{k-1}[3]) = 0$ and

$$\mathsf{H}(C_k[-n+2]) = \cdots = \mathsf{H}(C_{-n+1}) = \mathsf{H}(C_2[1]) = 0 \tag{7.15}$$

This means that $C_k[-n+1] \in \mathcal{X}_n^{\top} = \mathcal{X}$ and therefore $C_k \in \mathcal{X}[n-1]$. Hence $C_k[2] \in \mathcal{X}[n+1]$ and the exact sequence (7.10) becomes a short exact sequence

$$0 \longrightarrow \mathsf{H}(C_{k-1}[2]) \longrightarrow \mathsf{H}(X_{k-1}^*[n+1]) \longrightarrow \mathsf{H}(C_k[2]) \longrightarrow 0 \tag{7.16}$$

and the object $\mathsf{H}(C_k[2])$ is injective in $\mathsf{mod}\text{-}\mathcal{X}$. It follows that the exact sequence

$$0 \longrightarrow \mathsf{H}(X) \longrightarrow \mathsf{H}(X_0^*[n+1]) \longrightarrow \mathsf{H}(X_1^*[n+1]) \longrightarrow \cdots \longrightarrow \mathsf{H}(X_{k-1}^*[n+1]) \longrightarrow \mathsf{H}(C_k[2]) \longrightarrow 0$$

is an injective coresolution of $\mathsf{H}(X)$ and therefore $\mathsf{id}\,\mathsf{H}(X) \leqslant k$. We conclude that $\mathsf{silp}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$. $\square$

Finally the next result completes the proof of Theorem 7.5.

Proposition 7.9. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Let $0 \leqslant k \leqslant n-1$ and assume that $\mathcal{X}$ is strictly $(n-k)$-strong. If $n \geqslant 2k-1$, then:* $\mathsf{G\text{-}dim}\,\mathsf{mod}\text{-}\mathcal{X} = k$.

Proof. Let $X \in \mathcal{X}$. By Proposition 7.7, we have $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant k$. Assuming that $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant k-1$, $\forall X \in \mathcal{X}$, we show that $\mathcal{X}$ is $(n-k+1)$-strong. If $0 \leqslant k \leqslant 1$, then the assertion is clear. So assume that $k \geqslant 2$. We consider triangles $(T_t) : A^t \longrightarrow X^{t-1} \longrightarrow A^{t-1} \longrightarrow A^t[1]$, where each map $X^{t-1} \longrightarrow X^{t-1}$ is a right $\mathcal{X}$-approximation, $t \geqslant 1$, and $A^0 = X[n+1]$, so that $A^t = \Omega^t_{\mathcal{X}}(X[n+1])$. Applying H to the triangle (T_1), we obtain an exact sequence $0 \longrightarrow \mathsf{H}(A^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow \mathsf{H}(X[n+1]) \longrightarrow 0$ and $\mathcal{T}(\mathcal{X}, A^1[-i]) = 0$, $1 \leqslant i \leqslant n-k$. Using this and applying H to the triangle (T_2), we obtain an exact sequence $0 \longrightarrow \mathsf{H}(A^2) \longrightarrow \mathsf{H}(X^1) \longrightarrow \mathsf{H}(A^1) \longrightarrow 0$ and $\mathcal{T}(\mathcal{X}, A^2[-i]) = 0$, $1 \leqslant i \leqslant n-k-1$. Continuing in this way, we finally obtain an exact sequence $0 \longrightarrow \mathsf{H}(A^{k-1}) \longrightarrow \mathsf{H}(X^{k-2}) \longrightarrow \mathsf{H}(A^{k-2}) \longrightarrow 0$ and $\mathcal{T}(\mathcal{X}, A^{k-1}[-i]) = 0$, $1 \leqslant i \leqslant n-2k+2$. Since we assumed that $\mathsf{pd}\,\mathsf{H}(X[n+1]) \leqslant k-1$, the object $\mathsf{H}(A^{k-1})$ is projective. Hence there is a map $\alpha : X^* \longrightarrow A^{k-1}$, where $X^* \in \mathcal{X}$, inducing an isomorphism $\mathsf{H}(\alpha) : \mathsf{H}(X^*) \xrightarrow{\cong} \mathsf{H}(A^{k-1})$. Let $X^* \longrightarrow A^{k-1} \longrightarrow B \longrightarrow X^*[1]$ be a triangle. Applying H to this triangle and using that $\mathcal{X}$ is $(n-k)$-strong, the fact that $\mathsf{H}(\alpha)$ is invertible, and the vanishing condition $\mathcal{T}(\mathcal{X}, A^{k-1}[-i]) = 0$, $1 \leqslant i \leqslant n-2k+2$, we infer that $\mathcal{T}(\mathcal{X}, B[-k+1] = \mathcal{T}(\mathcal{X}, B[-k+2] = \cdots = \mathcal{T}(\mathcal{X}, B[-k+n]) = 0$, i.e. $B[-k] \in \mathcal{X}_n^{\top} = \mathcal{X}$. Hence $B \in \mathcal{X}[k]$ and this implies that the map $B \longrightarrow X^*[1]$ is zero since it lies in $\mathcal{T}(\mathcal{X}, \mathcal{X}[-k+1]$ and this space is zero since $1 \leqslant k-1 \leqslant n-k$ and $\mathcal{X}$ is $(n-k)$-strong and $n \geqslant 2k-1$. We infer that A^{k-1} admits a direct sum decomposition $A^{k-1} \cong X^* \oplus X'[k]$. On the other hand, since $A^{k-1} = \Omega^{k-1}_{\mathcal{X}}(X[n+1])$, we know by Remark 2.2 that A^{k-1} lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$. Since clearly any map from an object of $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k-1]$ to an object from $\mathcal{X}[k]$ is zero, it follows that the projection $A^{k-1} \longrightarrow X'[k]$ is zero and this implies that $A^{k-1} \cong X^* \in \mathcal{X}$. Then A^{k-2} lies in $\mathcal{X} \star \mathcal{X}[1]$ and using that $A^i = \Omega^i_{\mathcal{X}}(X[n+1])$, $\forall i \geqslant 1$, it follows inductively that $A^1 \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[k-2]$. Then $X[n+1]$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[k-1]$, hence $\mathcal{X}[n+1] \subseteq \mathcal{X} \star \mathcal{X}[1] \star \cdots \mathcal{X}[k-1]$. Then Proposition 7.3 shows that $\mathcal{X}$ is $(n-k+1)$-strong as required. $\square$

Remark 7.10. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Then Theorem 7.5 gives the following picture:

- If $\mathcal{X}$ is n-strong, then: $\mathsf{G\text{-}dim}\,\mathsf{mod}\text{-}\mathcal{X} = 0$.
- If $\mathcal{X}$ is strictly $(n-1)$-strong, then: $\mathsf{G\text{-}dim}\,\mathsf{mod}\text{-}\mathcal{X} = 1$.
- If $\mathcal{X}$ is strictly $(n-2)$-strong and $n \geqslant 3$, then: $\mathsf{G\text{-}dim}\,\mathsf{mod}\text{-}\mathcal{X} = 2$.

- If $\mathcal{X}$ is strictly $(n-3)$-strong and $n \geqslant 5$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = 3$.

  $\vdots$
- If $\mathcal{X}$ is strictly $(n-k)$-strong and $n \geqslant 2k-1$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = k$.

  $\vdots$
- If $\mathcal{X}$ is strictly 3-strong and $n \leqslant 7$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = n-3 \leqslant 4$.
- If $\mathcal{X}$ is strictly 2-strong and $n \leqslant 5$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = n-2 \leqslant 3$.
- If $\mathcal{X}$ is strictly 1-strong and $n \leqslant 3$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} = n-1 \leqslant 2$.
- If $n = 1$, then: $\mathsf{G\text{-}dim\,mod\text{-}}\mathcal{X} \leqslant 1$.

Since for an abelian category $\mathscr{A}$ we have $\mathsf{G\text{-}dim}\mathscr{A} \leqslant \mathsf{gl.\,dim}\,\mathscr{A}$, with equality if $\mathsf{gl.\,dim}\,\mathscr{A} < \infty$, we have the following consequence.

Corollary 7.11. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 1$. Let $0 \leqslant k \leqslant n-1$ and assume that $\mathcal{X}$ is $(n-k)$-strong. If $n \geqslant 2k-1$, then either $\mathsf{gl.\,dim\,mod\text{-}}\mathcal{X} = \infty$ or else $\mathsf{gl.\,dim\,mod\text{-}}\mathcal{X} \leqslant k$. Moreover if $\mathsf{gl.\,dim\,mod\text{-}}\mathcal{X} < \infty$ and $\mathcal{X}$ is strictly $(n-k)$-strong, then $\mathsf{gl.\,dim\,mod\text{-}}\mathcal{X} = k$.*

Corollary 7.12. *Let $\mathcal{X}$ be an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $0 \leqslant k \leqslant n-1$. Assume that $n \geqslant 2k-1$. Then $\mathsf{mod\text{-}}\mathcal{X}$ is Frobenius if and only if $\mathcal{T}(\mathcal{X}, \mathcal{X}[-n+k-i]) = 0$, $1 \leqslant i \leqslant k$.*

The next result characterizes strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ in terms of vanishing of the obstructions groups $\mathcal{O}_{-,-}$.

Proposition 7.13. *Let $\mathcal{X}$ be a $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 2$. Then for $1 \leqslant k \leqslant n-1$, the following statements are equivalent:*

(i) *$\mathcal{X}$ is $(n-k)$-strong.*
(ii) *$\mathcal{O}_{\mathcal{X}[-i],-} = 0$, $1 \leqslant i \leqslant n-k$.*

In particular if the functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod\text{-}}\mathcal{X}$ is full, then $\mathcal{X}$ is $(n-k)$-strong, for any k with $1 \leqslant k \leqslant n-1$, and $\mathsf{mod\text{-}}\mathcal{X}$ is 1-Gorenstein.

Proof. (i) ⇒ (ii) Since $\mathcal{X}$ is $(n-k)$-strong, we have $\mathsf{H}(\mathcal{X}[-i]) = \mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, for $1 \leqslant i \leqslant n-k$. This clearly implies that $\mathcal{O}_{\mathcal{X}[-i],-} = 0$, $1 \leqslant i \leqslant n-k$.

(ii) ⇒ (i) Assume that $\mathcal{O}_{\mathcal{X}[-i],-} = 0$, $1 \leqslant i \leqslant n-k$, i.e. the maps $\mathcal{T}(X[-i]), B) \longrightarrow \mathsf{Hom}(\mathsf{H}(X[-i]), \mathsf{H}(B))$ are surjective, $\forall B \in \mathcal{T}$, $\forall X \in \mathcal{X}$. For $X \in \mathcal{X}$, consider the triangle

$$\Omega^n_{\mathcal{X}}(X[-1])[n-1] \longrightarrow \mathsf{Cell}_{n-1}(X[-1]) \longrightarrow X[-1] \longrightarrow \Omega^n_{\mathcal{X}}(X[-1])[n]$$

arising from the cellular tower of $X[-1]$. Then we know that $\Omega^n_{\mathcal{X}}(X[-1]) \in \mathcal{X}$ and $\mathsf{Cell}_{n-1}(X[-1]) \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n-1]$. Setting $X^* := \Omega^n_{\mathcal{X}}(X[-1]) \in \mathcal{X}$ and $C := \mathsf{Cell}_{n-1}(X[-1])[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$, we have, by Proposition 4.2, that $\mathsf{H}(C) = 0$ and moreover there is a triangle

$$X^*[n] \xrightarrow{\beta} C \xrightarrow{\alpha} X \xrightarrow{\gamma} X^*[n+1] \tag{T}$$

Applying H to the triangle (T) and using that $\mathsf{H}(X^*[i]) = 0$, $1 \leqslant i \leqslant n$, we get isomorphisms in $\mathsf{mod\text{-}}\mathcal{X}$:

$$\mathsf{H}(\alpha[-i]) \ : \ \mathsf{H}(C[-i]) \xrightarrow{\cong} \mathsf{H}(X[-i]), \quad 1 \leqslant i \leqslant n-1 \tag{7.17}$$

Since $\mathcal{O}_{X[-i],C[-i]} = 0$, $1 \leqslant i \leqslant n-k$, there are maps $\delta_i : X[-i] \longrightarrow C[-i]$ in $\mathcal{T}$, $1 \leqslant i \leqslant n-k$ such that:

$$\mathsf{H}(\delta_i) = \mathsf{H}(\alpha[-i])^{-1} \ : \ \mathsf{H}(X[-i]) \xrightarrow{\cong} \mathsf{H}(C[-i]), \quad 1 \leqslant i \leqslant n-k \tag{7.18}$$

Then the map $\delta_i[i] : X \longrightarrow C$ lies in $\mathcal{T}(X, C)$ and this is zero since $\mathsf{H}(C) = 0$. Hence $\delta_i[i] = 0$ and therefore $\delta_i = 0$, $1 \leqslant i \leqslant n-k$. Then the isomorphisms $\mathsf{H}(\delta_i)$ are zero and this implies that $\mathsf{H}(C[-i]) \cong \mathsf{H}(X[-i]) = 0$, i.e. $\mathcal{T}(\mathcal{X}, X[-i]) = 0$, $1 \leqslant i \leqslant n-k$. Since X is an arbitrary object of $\mathcal{X}$, we infer that $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant n-k$ and consequently the $(n+1)$-cluster tilting subcategory $\mathcal{X}$ is $(n-k)$-strong.

If H is full, then $\mathcal{O}_{?,-} = 0$, in particular $\mathcal{O}_{\mathcal{X}[-i],-} = 0$, $1 \leqslant i \leqslant n-1$. Then by (i), $\mathcal{X}$ is $(n-1)$-strong and therefore $\mathsf{mod\text{-}}\mathcal{X}$ is 1-Gorenstein by Theorem 7.5. □

7.4. **Keller-Reiten's Morita Theorem for the $(n+1)$-Cluster category.** We use the results of subsection 7.3 to give an application to the characterization of the $(n+1)$-cluster category of a simply laced Dynkin quiver due to Keller-Reiten [22]. First we recall some definitions which also will be used later on.

Assume that the triangulated category $\mathcal{T}$ is k-linear over a field k, all Hom-spaces are finite-dimensional, and admits a **Serre functor** $\mathbb{S}$ [13]. Thus $\mathbb{S}$ is an triangulated auto-equivalence of $\mathcal{T}$, and for any objects A, B in $\mathcal{T}$ there are natural isomorphisms

$$\mathsf{D}\,\mathsf{Hom}_{\mathcal{T}}(A,B) \xrightarrow{\cong} \mathsf{Hom}_{\mathcal{T}}(B,\mathbb{S}A)$$

where D denotes the k-dual functor. If $\mathcal{T}$ admits a Serre functor $\mathbb{S}$, then $\mathcal{T}$ is called (**weakly**) d-**Calabi-Yau**, for some $d \geqslant 1$, provided that $\mathbb{S}(?) \cong (?)[d]$ as (additive) triangulated functors.

Let k be a field and H a finite-dimensional hereditary k-algebra. Then it is well-known that the bounded derived category $\mathbf{D}^b(\mathsf{mod}\text{-}H)$ of finite-dimensional H-modules admits a Serre functor ν. If $d \geqslant 1$ is an integer, then the d-**cluster category** $\mathscr{C}_H^{(d)}$ of H, as defined by Keller [20], is the orbit category $\mathscr{C}_H^{(d)} := \mathbf{D}^b(\mathsf{mod}\text{-}H)/(\nu^{-1}[d])^{\mathbb{Z}}$ of $\mathbf{D}^b(\mathsf{mod}\text{-}H)$ under the action of the automorphism group generated by $X^\bullet \mapsto \nu^{-1}(X^\bullet[d])$. If $H = kQ$ is the path algebra of a quiver Q, then we say that $\mathscr{C}_H^{(d)}$ is the d-cluster category of the quiver Q and we write $\mathscr{C}_Q^{(d)}$. As shown by Keller [20], the d-cluster category $\mathscr{C}_H^{(d)}$ is a d-Calabi-Yau triangulated category and the projection functor $\pi : \mathbf{D}^b(\mathsf{mod}\text{-}kQ) \longrightarrow \mathscr{C}_H^{(d)}$ is triangulated.

Recall that a triangulated category $\mathcal{T}$ is *algebraic* if $\mathcal{T}$ is triangle equivalent to the stable category of a Frobenius category.

Theorem 7.14. [22] *Let $\mathcal{T}$ be a k-linear triangulated category with finite-dimensional* Hom*-spaces over an algebraically closed field k. Then for an integer $n \geqslant 1$, the following statements are equivalent.*

(i) *$\mathcal{T}$ is triangle equivalent to the $(n+1)$-cluster category $\mathscr{C}_Q^{(n+1)}$ of some simply laced Dynkin quiver Q.*
(ii) *$\mathcal{T}$ is algebraic $(n+1)$-Calabi-Yau and admits a $(n-1)$-strong $(n+1)$-cluster tilting object T such that the endomorphism algebra $\mathsf{End}_{\mathcal{T}}(T)$ has finite global dimension.*

Proof. (i) $\Rightarrow$ (ii) If condition (i) holds, then by Keller [20] we know that $\mathscr{C}_Q^{(n+1)}$ is algebraic $(n+1)$-Calabi-Yau and it is shown by Keller-Reiten in [21] that $\pi(H)$ is a $(n+1)$-cluster tilting object in $\mathscr{C}_Q^{(n+1)}$, where $H = kQ$ considered as a stalk complex in $\mathbf{D}^b(\mathsf{mod}\text{-}H)$. Moreover in [22] it is shown that the $(n+1)$-cluster tilting object $\pi(H)$ is $(n-1)$-strong and its endomorphism algebra $\mathsf{End}(\pi(H))$ has finite global dimension.

(ii) $\Rightarrow$ (i) Conversely if condition (ii) holds, then by Theorem 7.5 we have that $\mathsf{End}_{\mathcal{T}}(T)$ is 1-Gorenstein and then by Corollary 7.11, $\mathsf{End}_{\mathcal{T}}(T)$ is hereditary. Setting $H = \mathsf{End}_{\mathcal{T}}(T)$, by Theorem 4.2 in Keller-Reiten [22], there is a triangle equivalence $F : \mathcal{T} \xrightarrow{\approx} \mathscr{C}_H^{(n+1)}$ such that $F(T) = \pi(H)$. $\square$

7.5. **Abelian Subcategories.** Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. We show that if the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ has finite global dimension, and $\mathcal{X}$ is $(n-k)$-strong, for some $0 \leqslant k \leqslant \frac{n}{2}$, then $\mathsf{mod}\text{-}\mathcal{X}$ can be realized as a full subcategory of $\mathcal{T}$, in some cases via a ∂-functor.

For an abelian category $\mathcal{M}$ we denote by $\mathsf{Proj}^{\leqslant k}\,\mathcal{M}$, resp. $\mathsf{Proj}^{<\infty}\,\mathcal{M}$, the full subcategory of $\mathcal{M}$ consisting of the objects with projective dimension $\leqslant k$, resp. $< \infty$.

Proposition 7.15. *Let $\mathcal{X}$ be a contravariantly finite t-rigid subcategory of $\mathcal{T}$, $t \geqslant 1$, and assume that $\mathcal{X}$ is $(t-1)$-strong, if $t \geqslant 2$. Then the functor $\mathsf{H} : \mathcal{T} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ induces a full embedding*

$$\mathsf{H} \;\; : \;\; \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\big) \cap \mathcal{X}_t^{\top}[t+1] \;\; \longrightarrow \;\; \mathsf{Proj}^{\leqslant t}\,\mathsf{mod}\text{-}\mathcal{X}$$

which is an equivalence if $\mathcal{X}$ is t-strong.

Proof. Set $\mathcal{U}_t := \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\big) \cap \mathcal{X}_t^{\top}[t+1]$ and $\mathsf{H}_t := \mathsf{H}|_{\mathcal{U}_t}$. First note that by Theorem 5.4, we have $\mathsf{pd}\,\mathsf{H}(A) \leqslant t$, for any object $A \in \mathcal{U}_t$, so we have a functor $\mathsf{H}_t : \mathcal{U}_t \longrightarrow \mathsf{Proj}^{\leqslant k}\,\mathsf{mod}\text{-}\mathcal{X}$.

Let $\alpha : A \longrightarrow B$ be a map in $\mathcal{U}_t$ such that $\mathsf{H}(\alpha) = 0$. Then α factorizes through the map $h_A^0 : A \longrightarrow \Omega_{\mathcal{X}}^1(A)[1]$, say via a map $\beta : \Omega_{\mathcal{X}}^1(A)[1] \longrightarrow B$. Since $\mathcal{X}$ is t-rigid and A lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$, it follows that $\Omega_{\mathcal{X}}^t(A)$ lies in $\mathcal{X}$ and then it is easy to see that $\Omega_{\mathcal{X}}^1(A)$ lies in $\mathcal{X} \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t-1]$, hence $\Omega_{\mathcal{X}}^t(A)[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t]$. Since $B \in \mathcal{X}_t^{\top}[t+1]$, we have $\mathcal{T}(\mathcal{X}, B[-i]) = 0$, $1 \leqslant i \leqslant t$. Since, as easily seen, any map from an object form $\mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t]$ to an object in $\mathcal{X}_t^{\top}[t+1]$ is zero, we have $\beta = 0$. Therefore $\alpha = 0$ and H_t is faithful. Next let $\alpha : \Omega_{\mathcal{X}}^1(A) \longrightarrow B$ be an $\mathcal{X}$-ghost map. Then α factorizes through the map $h_A^1 : \Omega_{\mathcal{X}}^1(A) \longrightarrow \Omega_{\mathcal{X}}^2(A)[1]$,, say via a map $\beta : \Omega_{\mathcal{X}}^2(A)[1] \longrightarrow B$. As above, $\Omega_{\mathcal{X}}^2(A)$ lies in $\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-2]$, hence $\Omega_{\mathcal{X}}^2(A)[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t-1]$. Since $B \in \mathcal{X}_t^{\top}[t+1]$, it follows directly that any map from an object form $\mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[t-1]$ to an object in $\mathcal{X}_t^{\top}[t+1]$ is zero, hence $\beta = 0$, and therefore $\alpha = 0$. Consequently $\mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A), B) = 0$, and therefore the obstruction group $\mathcal{O}_{A,B}$ vanishes. Then from Proposition 3.6, we infer that the map $\mathcal{T}(A, B) \longrightarrow \mathsf{Hom}(\mathsf{H}(A), \mathsf{H}(B)$ is surjective, hence $\mathsf{H}(\alpha) = \tilde{\alpha}$ for some map $\alpha : A \longrightarrow B$. It follows that H_t is full.

Now consider an object $F \in \mathsf{mod}\text{-}\mathcal{X}$, with $\mathsf{pd}\, F \leqslant t$. Then $\mathsf{H}(A) = F$, and we may choose $A \in \mathcal{X} \star \mathcal{X}[1]$. We have a projective resolutions $0 \longrightarrow \mathsf{H}(X^t_A) \longrightarrow \mathsf{H}(X^{t-1}) \longrightarrow \cdots \longrightarrow \mathsf{H}(X^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathcal{X}$. The last map of the resolution gives us a map $X^t \longrightarrow X^{t-1}$ which induces a triangle $X^t \longrightarrow X^{t-1} \longrightarrow A^{t-1} \longrightarrow X^t[1]$. Applying H to this triangle and using that $\mathcal{X}$ is $(t-1)$-strong, so $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant t-1$, we see easily that $\mathcal{T}(\mathcal{X}, A^{t-1}[-i]) = 0$, $1 \leqslant i \leqslant t-1$, and moreover $\mathsf{Im}\,\big(\mathsf{H}(X^{t-1}) \longrightarrow \mathsf{H}(X^{t-2})\big) = \mathsf{H}(A^{t-1})$. Consider the induced monomorphism $\mathsf{H}(A^{t-1}) \longrightarrow \mathsf{H}(X^{t-2})$. Since $X^t \in \mathcal{X}$, we have $A^{t-1} \in \mathcal{X} \star \mathcal{X}[1]$ and $\Omega^1_{\mathcal{X}}(A^{t-1}) = X^t \in \mathcal{X}$. Since $\mathcal{X}$ is $(t-1)$-strong, by Lemma 6.4, we have $\mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A^{t-1}), X^{t-2}) = 0$ and therefore $\mathcal{O}_{A^{t-1}, X^{t-2}} = 0$. It follows that the monomorphism $\mathsf{H}(A^{t-1}) \longrightarrow \mathsf{H}(X^{t-2})$ is induced by a map $A^{t-1} \longrightarrow X^{t-2}$. Consider a triangle $A^{t-1} \longrightarrow X^{t-2} \longrightarrow A^{t-2} \longrightarrow A^{t-1}[1]$. As above, applying H to this triangle we see easily that $\mathcal{T}(\mathcal{X}, A^{t-2}[-i]) = 0$, $1 \leqslant i \leqslant t-1$, and moreover $\mathsf{Im}\,\big(\mathsf{H}(X^{t-2}) \longrightarrow \mathsf{H}(X^{t-3})\big) = \mathsf{H}(A^{t-2})$. Moreover we have $A^{t-2} \in \mathcal{X} \star \mathcal{X}[1] \star \mathcal{X}[2]$ and $\Omega^1_{\mathcal{X}}(A^{t-2}) = A^{t-1} \in \mathcal{X} \star \mathcal{X}[1]$. By Lemma 6.4 we have $\mathsf{Gh}_{\mathcal{X}}(\Omega^1_{\mathcal{X}}(A^{t-2}), X^{t-3}) = 0$, so $\mathcal{O}_{A^{t-2}, X^{t-3}} = 0$. We infer that the monomorphism $\mathsf{H}(A^{t-2}) \longrightarrow \mathsf{H}(X^{t-3})$ is induced by a map $A^{t-2} \longrightarrow X^{t-3}$. Considering a triangle $A^{t-2} \longrightarrow X^{t-3} \longrightarrow A^{t-3} \longrightarrow A^{t-2}[1]$ and continuing in this way, we construct triangles $A^j \longrightarrow X^{j-1} \longrightarrow A^{j-1} \longrightarrow A^j[1]$, with $X^j \in \mathcal{X}$, and exact sequences $0 \longrightarrow \mathsf{H}(A^j) \longrightarrow \mathsf{H}(X^{j-1}) \longrightarrow \mathsf{H}(A^{j-1}) \longrightarrow 0$, for $1 \leqslant j \leqslant t$, where $A^t = X^t$, and the objects A^j satisfy $\mathcal{T}(\mathcal{X}, A^j[-i]) = 0$, $1 \leqslant i \leqslant t-1$. In particular for $j = 1$, we have a triangle $A^1 \longrightarrow X^0 \longrightarrow A^0 \longrightarrow A^1[1]$, where $A^0 \in \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]$ and $\mathcal{T}(\mathcal{X}, A^0[-i]) = 0$. $1 \leqslant i \leqslant t-1$, and this implies that $\mathsf{H}(A^0) \cong \mathsf{H}(A) \cong F$. Hence F is isomorphic to an object $\mathsf{H}(A^0)$, where A^0 lies in $\big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t]\big) \cap \mathcal{X}^{\top}_{t-1}[t]$. Finally assume that $\mathcal{X}$ is t-strong, i.e. we have in addition $\mathcal{T}(\mathcal{X}, \mathcal{X}[-t]) = 0$. Then applying H to the triangle $A^1 \longrightarrow X^0 \longrightarrow A^0 \longrightarrow A^1[1]$ and using that $A^1, A^0 \in \mathcal{X}^{\top}_{t-1}[t]$, we have directly that $\mathsf{H}(A^0[-t]) = 0$, so A^0 lies in $\mathcal{X}^{\top}_t[t+1]$ and therefore $A^0 \in \mathcal{U}_t$. This shows that $\mathsf{H} : \mathcal{U}_t \longrightarrow \mathsf{Proj}^{\leqslant t}\,\mathsf{mod}\text{-}\mathcal{X}$ is essentially surjective. $\square$

Theorem 7.16. *Let $\mathcal{X}$ be an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $0 \leqslant k \leqslant n-1$.*

(i) *If $\mathsf{mod}\text{-}\mathcal{X}$ has finite global dimension, and $n \geqslant 2k$, then $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$ and there is an equivalence*

$$\mathsf{H} \;:\; \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\big) \cap \mathcal{X}^{\top}_k[k+1] \;\xrightarrow{\approx}\; \mathsf{mod}\text{-}\mathcal{X}$$

(ii) *If $k = 1$, then the induced full embedding $\mathsf{T} : \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathcal{T}$ is a ∂-functor, which extends uniquely to an additive functor $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{X}) \longrightarrow \mathcal{T}$ commuting with the shifts.*

Proof. Under the imposed assumptions, as in the proof of Proposition 7.7, we see that $\mathcal{X}$ is $(k-1)$-strong. By Proposition 7.15, the functor H induces a full embedding $\mathsf{H}_k : \big(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[k]\big) \cap \mathcal{X}^{\top}_k[k+1] \longrightarrow \mathsf{Proj}^{\leqslant k}\,\mathsf{mod}\text{-}\mathcal{X}$. Since, by Theorem 7.5, $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein, finiteness of $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{X}$ implies that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant k$, and therefore $\mathsf{mod}\text{-}\mathcal{X} = \mathsf{Proj}^{\leqslant k}\,\mathsf{mod}\text{-}\mathcal{X}$. On the other hand since $n \geqslant 2k$, it follows that $\mathcal{X}^{\top}_{n-k}[n-k+1] \subseteq \mathcal{X}^{\top}_k[k+1]$, i.e. $\mathcal{X}$ is k-strong. Then by Proposition 7.15, the functor H_k is essentially surjective, hence an equivalence. Finally assume that $k = 1$, so $n \geqslant 2$, $\mathcal{X}$ is $(n-1)$-strong, and $\mathsf{mod}\text{-}\mathcal{X}$ is hereditary. By Corollary 3.9 we have an isomorphism $\mathcal{T}(A, B[1]) \cong \mathsf{Ext}^1(\mathsf{H}(A), \mathsf{H}(B))$, $\forall A, B \in \big(\mathcal{X} \star \mathcal{X}[1]\big) \cap \mathcal{X}^{\top}_1[2]$. It follows easily from this that the induced fully faithful functor $\mathsf{T}\colon \mathsf{mod}\text{-}\mathcal{X} \approx \big(\mathcal{X} \star \mathcal{X}[1]\big) \cap \mathcal{X}^{\top}_1[2] \longrightarrow \mathcal{T}$ is a ∂-functor; details are left to the reader. By a result of Amiot [1], T extends uniquely to an additive functor $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{X}) \longrightarrow \mathcal{T}$ commuting with the shift functors. $\square$

Note that the case $n = k = 1$, where the strongness condition is vacuous, the functor T gives the full embedding $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathcal{X} \approx \mathcal{X}[2] \longrightarrow \mathcal{T}$.

8. Gorenstein-Projectives

Our aim in this section is to investigate the full subcategory of Gorenstein-projective objects of the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$, where $\mathcal{X}$ is a $(n-1)$-strong $(n+1)$-cluster tilting subcategory.

For convenience, we call in this section $(n-1)$-strong $(n+1)$-cluster tilting subcategories simply strong $(n+1)$-cluster tilting subcategories.

8.1. Gorenstein-Projectives.

To proceed further it is convenient to have a description of the full subcategories of Gorenstein-projective and Gorenstein-injective objects of the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$.

Let $\mathscr{A}$ be an abelian category. A complex of projective objects $P^{\bullet} : \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ is called **totally acyclic** if $P^{\bullet}$ and the induced complex $\mathscr{A}(P^{\bullet}, Q)$ are acyclic, for any projective object Q of $\mathscr{A}$. Dually a complex of injective objects $I^{\bullet} : \cdots \longrightarrow I^{-1} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ is called **totally acyclic** if $I^{\bullet}$ and the induced complex $\mathscr{A}(J, I^{\bullet})$ are acyclic for any injective object J of $\mathscr{A}$.

Definition 8.1. (i) An object $G \in \mathscr{A}$ is called **Gorenstein-projective** if $G \cong \mathsf{Coker}(P^{-1} \longrightarrow P^0)$ for some totally acyclic complex $P^\bullet$ of projectives.

(ii) An object $G \in \mathscr{A}$ is called **Gorenstein-injective** if $G \cong \mathsf{Ker}(I^0 \longrightarrow I^1)$ for some totally acyclic complex $I^\bullet$ of injectives.

The full subcategory of Gorenstein-projective, resp. Gorenstein-injective, objects of $\mathscr{A}$ is denoted by $\mathsf{GProj}\,\mathscr{A}$, resp. $\mathsf{GInj}\,\mathscr{A}$. Also we denote by $\underline{\mathsf{GProj}}\,\mathscr{A}$, resp. $\overline{\mathsf{GInj}}\,\mathscr{A}$, the stable category of $\mathsf{GProj}\,\mathscr{A}$, resp. $\mathsf{GInj}\,\mathscr{A}$, modulo projectives, resp. injectives.

In the following remark we remind the reader of basic properties of Gorenstein categories which will be used in the sequel. We refer to [12, 7], for more detailed discussions.

Remark 8.2. Let $\mathscr{A}$ be an abelian category with enough projective and/or injective objects.

• For any objects $G_1 \in \mathsf{GProj}\,\mathscr{A}$, $G_2 \in \mathsf{GInj}\,\mathscr{A}$, $A \in \mathscr{A}$, and any $k \geqslant 1$ there are isomorphisms:

$$\mathsf{Ext}^k(G_1, A) \xrightarrow{\cong} \underline{\mathsf{Hom}}(\Omega^k G_1, A) \qquad \text{and} \qquad \mathsf{Ext}^k(A, G_2) \xrightarrow{\cong} \underline{\mathsf{Hom}}(A, \Sigma^k G_2)$$

• The categories $\mathsf{GProj}\,\mathscr{A}$ and $\mathsf{GInj}\,\mathscr{A}$ are Frobenious exact subcategories of $\mathscr{A}$. Hence the stable categories $\underline{\mathsf{GProj}}\,\mathscr{A}$ and $\overline{\mathsf{GInj}}\,\mathscr{A}$ are triangulated.

• If $\mathscr{A}$ is Gorenstein then $\mathsf{Proj}^{<\infty}\,\mathscr{A} = \mathsf{Inj}^{<\infty}\,\mathscr{A}$; if $\mathscr{A}$ is of Goresntein dimension $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} = d$, then $\mathsf{Proj}^{<\infty}\,\mathscr{A} = \mathsf{Proj}^{\leqslant d}\,\mathscr{A}$ (and dually $\mathsf{Inj}^{<\infty}\,\mathscr{A} = \mathsf{Inj}^{\leqslant d}\,\mathscr{A}$). Moreover we have $\mathsf{GProj}\,\mathscr{A} = \Omega^d\mathscr{A}$ and $\mathsf{GInj}\,\mathscr{A} = \Sigma^d\mathscr{A}$, see [7, Theorem 4.16]. It follows that if $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} \leqslant 1$, then $\mathsf{GProj}\,\mathscr{A}$ consists of the subobjects of the projective objects and $\mathsf{GInj}\,\mathscr{A}$ consists of the factors of the injectives.

• The inclusion functor $\underline{\mathsf{GProj}}\,\mathscr{A} \longrightarrow \underline{\mathscr{A}}$ admits as a right adjoint the functor $\Omega^{-d}\Omega^d : \underline{\mathscr{A}} \longrightarrow \underline{\mathsf{GProj}}\,\mathscr{A}$. Dually the inclusion functor $\overline{\mathsf{GInj}}\,\mathscr{A} \longrightarrow \overline{\mathscr{A}}$ admits as a left adjoint the functor $\Sigma^{-d}\Sigma^d : \overline{\mathscr{A}} \longrightarrow \overline{\mathsf{GInj}}\,\mathscr{A}$.

Let A be in $\mathfrak{T}$. Then there exists a triangle

$$\Omega^n_{\mathfrak{X}}(A[n])[-1] \longrightarrow \mathsf{Cell}_{n-1}(A[n])[-n] \longrightarrow A \longrightarrow \Omega^n_{\mathfrak{X}}(A[n]) \tag{8.1}$$

where the last map $\omega^{n-1}_{A[n]}[-n]\colon A \longrightarrow \Omega^n_{\mathfrak{X}}(A[n])$ is a left $\mathfrak{X}$-approximation of A and

$$\mathsf{Cell}_{n-1}(A[n])[-n] \in \mathfrak{X}[-n] \star \mathfrak{X}[-n+1] \star \cdots \star \mathfrak{X}[-1]$$

Lemma 8.3. *Assume that the $(n+1)$-cluster tilting subcategory $\mathfrak{X}$ is strong. Then the full subcategory* $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathfrak{X}$, *resp.* $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathfrak{X}$, *of projective, resp. injective, objects of* $\mathsf{mod}\text{-}\mathfrak{X}$ *is functorially finite in* $\mathsf{mod}\text{-}\mathfrak{X}$. *Moreover for any object $A \in \mathfrak{T}$, the map*

$$\mathsf{H}(\omega^{n-1}_{A[n]}[-n]) : \mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^n_{\mathfrak{X}}(A[n])\big)$$

is a left projective approximation of $\mathsf{H}(A)$, *and the object* $\mathsf{H}(A)$ *is Gorenstein-projective if and only if the map* $\mathsf{H}(\omega^{n-1}_{A[n]}[-n])$ *is a monomorphism.*

Proof. We know that $\mathsf{mod}\text{-}\mathfrak{X}$ has enough projective and enough injective objects. Hence $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathfrak{X}$ is contravariantly finite and $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathfrak{X}$ is covariantly finite. Let $F = \mathsf{H}(A) \in \mathsf{mod}\text{-}\mathfrak{X}$. We claim that the map $\mathsf{H}(\omega^{n-1}_{A[n]}[-n]) : \mathsf{H}(A) \longrightarrow \mathsf{H}(\Omega^n_{\mathfrak{X}}(A[n]))$ induced by the triangle (8.1) is a left projective approximation of $\mathsf{H}(A)$. Indeed $\mathsf{H}(\Omega^n_{\mathfrak{X}}(A[n]))$ is projective since $\Omega^n_{\mathfrak{X}}(A[n]) \in \mathfrak{X}$. Let $\alpha : \mathsf{H}(A) \longrightarrow \mathsf{H}(X)$ be a map, where $X \in \mathfrak{X}$. By applying the Octahedral axiom to the composition $\gamma^1_A \circ \omega^{n-1}_{A[n]}[-n] : \mathsf{Cell}_1(A) \longrightarrow A \longrightarrow \Omega^n_{\mathfrak{X}}(A[n])$, it is easy to see that there is triangle $\Omega^n_{\mathfrak{X}}(A[n])[-1] \longrightarrow B \longrightarrow \mathsf{Cell}_1(A) \longrightarrow \Omega^n_{\mathfrak{X}}(A[n])$, where B lies in $\mathfrak{X}[1] \star \cdots \star \mathfrak{X}[n-1] \star \mathfrak{X}[-n] \star \cdots \star \mathfrak{X}[-1]$. Now since $\mathsf{Cell}_1(A)$ lies in $\mathfrak{X} \star \mathfrak{X}[1]$, the composition $\mathsf{H}(\gamma^1_A) \circ \alpha : \mathsf{H}(\mathsf{Cell}_1(A)) \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(X)$ is of the form $\mathsf{H}(\beta)$ for some map $\beta : \mathsf{Cell}_1(A) \longrightarrow X$. Since the $(n+1)$-cluster tilting subcategory $\mathfrak{X}$ is strong, any map from an object in $\mathfrak{X}[1] \star \cdots \star \mathfrak{X}[n-1] \star \mathfrak{X}[-n] \star \cdots \star \mathfrak{X}[-1]$ to an object in $\mathfrak{X}$ is clearly zero, and therefore the composition of $B \longrightarrow \mathsf{Cell}_1(A)$ with β is zero. Hence there exists a map $\rho : \Omega^n_{\mathfrak{X}}(A[n]) \longrightarrow X$ such that $\beta = \gamma^1_A \circ \omega^{n-1}_{A[n]}[-n] \circ \rho$. Then $\mathsf{H}(\gamma^1_A) \circ \alpha = \mathsf{H}(\beta) = \mathsf{H}(\gamma^1_A) \circ \mathsf{H}(\omega^{n-1}_{A[n]}[-n]) \circ \mathsf{H}(\rho)$, and since $\mathsf{H}(\gamma^1_A)$ is invertible, we have $\alpha = \mathsf{H}(\omega^{n-1}_{A[n]}[-n]) \circ \mathsf{H}(\rho)$. This shows that $\mathsf{H}(\omega^{n-1}_{A[n]}[-n])$ is a left projective approximation of $\mathsf{H}(A)$ and $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathfrak{X}$ is covariantly finite in $\mathsf{mod}\text{-}\mathfrak{X}$. If $\mathsf{H}(\omega^{n-1}_{A[n]}[-n])$ is a monomorphism, then $\mathsf{H}(A)$ is Gorenstein-projective as a subobject of the projective object $\mathsf{H}(\Omega^n_{\mathfrak{X}}(A[n])$. Conversely if $\mathsf{H}(A)$ is Gorenstein-projective, then there is a monomorphism $\mu : \mathsf{H}(A) \longrightarrow \mathsf{H}(X)$, where $X \in \mathfrak{X}$. Since μ factorizes through $\mathsf{H}(\omega^{n-1}_{A[n]}[-n])$, it follows that the latter is a monomorphism.

The proof that $\mathsf{Inj}\,\mathsf{mod}\text{-}\mathfrak{X}$ is contravariantly finite is dual and is left to the reader. □

The following characterization of Gorenstein-projective functors will be useful later; its dual version concerning Gorenstein-injective functors is left to the reader.

Lemma 8.4. *Let $\mathcal{X}$ be a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Then for any object $A \in \mathcal{T}$ the following statements are equivalent.*

(i) $\mathsf{H}(A)$ *is Gorenstein-projective.*

(ii) *The map* $A \longrightarrow \Omega^n_{\mathcal{X}}(A[n])$ *induces a monomorphism* $\mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^n_{\mathcal{X}}(A[n])\big)$.

(iii) **Case $n \geqslant 2$:** $A \in ({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$, *or equivalently* $\mathsf{H}\big(\mathsf{Cell}_{n-1}(A[n])[-n]\big) = 0$.
Case $n = 1$: $\mathsf{Gh}_{\mathcal{X}}(\mathcal{X}[-1], A) = \mathcal{T}(\mathcal{X}[-1], A)$, *i.e. any map* $X[-1] \to A$, *with* $X \in \mathcal{X}$ *is* $\mathcal{X}$*-ghost.*

Proof. Since $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein, by Remark 8.2, an object $F = \mathsf{H}(A)$ in $\mathsf{mod}\text{-}\mathcal{X}$ is Gorenstein-projective if and only if F is a subobject of a projective object. Clearly this is equivalent to say that F admits a monomorphic left $(\mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X})$-approximation. Then the equivalence (i) $\Leftrightarrow$ (ii) follows from Corollary 8.3.

(ii) $\Leftrightarrow$ (iii) If $n \geqslant 2$, then applying $\mathsf{H}(A)$ to the triangle (8.1) we have a long exact sequence

$$\cdots \longrightarrow \mathsf{H}\big(\Omega^n_{\mathcal{X}}(A[n])[-1]\big) \longrightarrow \mathsf{H}\big(\mathsf{Cell}_{n-1}(A[n])[-n]\big) \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^n_{\mathcal{X}}(A[n])\big) \longrightarrow \cdots$$

Since $\Omega^n_{\mathcal{X}}(A[n]) \in \mathcal{X}$ it follows that $\mathsf{H}\big(\Omega^n_{\mathcal{X}}(A[n])[-1]\big) = 0$. Hence the map $\mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^n_{\mathcal{X}}(A[n])\big)$ is a monomorphism if and only if $\mathsf{H}\big(\mathsf{Cell}_{n-1}(A[n])[-n]\big) = 0$. Since always $\mathsf{Cell}_{n-1}(A[n])$ lies in $(\mathcal{X} \star \cdots \star \mathcal{X}[n-1])$ it follows that $\mathsf{Cell}_{n-1}(A[n])[-n]$ lies in $(\mathcal{X}[-n] \star \cdots \star \mathcal{X}[-1])$ which is equal to ${}^{\top}\mathcal{X}$ by Corollary 4.6. Hence $\mathsf{H}\big(\mathsf{Cell}_{n-1}(A[n])[-n]\big) = 0$ if and only if $\mathsf{Cell}_{n-1}(A[n])[-n]$ lie in $\mathsf{Ker}\,\mathsf{H} = \mathcal{X}^{\top}$, and therefore if and only if $\mathsf{Cell}_{n-1}(A[n])[-n]$ lies in ${}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}$. Using the triangle (8.1) this in turn is equivalent to $A \in ({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$. If $n = 1$, then the triangle (8.1) takes the form $X^1_{A[1]}[-1] \longrightarrow X^0_{A[1]}[-1] \longrightarrow A \longrightarrow X^1_{A[1]}$ and then $\mathsf{H}(A)$ is Gorenstein-projective iff the middle map is killed by H, equivalently the middle map factorizes through $\mathcal{X}^{\top}$. Since $n = 1$ we have $\mathcal{X}^{\top} = \mathcal{X}[1]$ and then clearly $\mathsf{H}(A)$ is Gorenstein-projective if and only if any map $X[-1] \longrightarrow A$, with $X \in \mathcal{X}$, is $\mathcal{X}$-ghost, i.e. factorizes through an object from $\mathcal{X}[1]$. $\square$

It is well known that over a d-Gorenstein abelian category $\mathscr{A}$ the full subcategory of Gorenstein-projectives is contravariantly finite and the full subcategory of objects with finite projective dimension which coincides with the full subcategory of objects with finite injective dimension, is functorially finite. The next result describes the Gorenstein-projective approximation and the left and right approximation by objects of finite projective dimension of any object of the 1-Gorenstein abelian category $\mathsf{mod}\text{-}\mathcal{X}$.

Proposition 8.5. *Let $\mathcal{X}$ be a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Then the full subcategory $({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$ is contravariantly finite in $\mathcal{T}$. More precisely for any object $A \in \mathcal{T}$, there exists a triangle*

$$\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big) \longrightarrow G_A \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1] \tag{8.2}$$

where $\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big) \in \mathcal{X}$, and the map $G_A \longrightarrow A$ is a right $({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$-approximation of A. Moreover the above triangle induces a short exact sequence

$$0 \longrightarrow \mathsf{H}\big(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)\big) \longrightarrow \mathsf{H}(G_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$$

and the map $\mathsf{H}(G_A) \longrightarrow \mathsf{H}(A)$ is a right Gorenstein-projective approximation of $\mathsf{H}(A)$.

Proof. Consider the triangle arising from the cellular tower of $A[-1]$.

$$\mathsf{Cell}_{n-1}(A[-1])[1] \longrightarrow A \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1] \longrightarrow \mathsf{Cell}_{n-1}(A[-1])[2]$$

Then we know that $\Omega^n_{\mathcal{X}}(A[-1])$ lies in $\mathcal{X}$. Since the injective object $\mathsf{H}(\Omega^n_{\mathcal{X}}(A[-1])[n+1])$ has projective dimension at most one, we have a triangle

$$\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big) \longrightarrow X^0_{\Omega^n_{\mathcal{X}}(A[-1])[n+1]} \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1] \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)$$

where the object $\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)$ lies in $\mathcal{X}$. Now form the weak-pull back of the first triangle along the map $X^0_{\Omega^n_{\mathcal{X}}(A[-1])[n+1]} \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1]$ in the sense of [8]:

$$\begin{tikzcd}
 & \Omega^1_{\mathcal{X}}(\Omega^n_{\mathcal{X}}(A[-1])[n+1]) \arrow[r, equal] \arrow[d] & \Omega^1_{\mathcal{X}}(\Omega^n_{\mathcal{X}}(A[-1])[n+1]) \arrow[d] & \\
\mathsf{Cell}_{n-1}(A[-1])[1] \arrow[r] \arrow[d, equal] & G_A \arrow[r] \arrow[d] & X^0_{\Omega^n_{\mathcal{X}}(A[-1])[n+1]} \arrow[r] \arrow[d] & \mathsf{Cell}_{n-1}(A[-1])[2] \arrow[d, equal] \\
\mathsf{Cell}_{n-1}(A[-1])[1] \arrow[r] & A \arrow[r] \arrow[d] & \Omega^n_{\mathcal{X}}(A[-1])[n+1] \arrow[r] \arrow[d] & \mathsf{Cell}_{n-1}(A[-1])[2] \\
 & \Omega^1_{\mathcal{X}}(\Omega^n_{\mathcal{X}}(A[-1])[n+1])[1] \arrow[r, equal] & \Omega^1_{\mathcal{X}}(\Omega^n_{\mathcal{X}}(A[-1])[n+1])[1] &
\end{tikzcd}$$

Since the object $\mathsf{Cell}_{n-1}(A[-1])[1]$ lies clearly in ${}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}$ it follows that G_A lies in $({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$. This implies, by Lemma 8.4, that the object $\mathsf{H}(G_A)$ is Gorenstein-projective. Applying the functor H to the above diagram of triangles and using that the maps $\mathsf{H}(G_A) \longrightarrow \mathsf{H}(X^0_{\Omega^n_{\mathcal{X}}(A[-1])[n+1]}) \longleftarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(\Omega^n_{\mathcal{X}}(A[-1])[n+1]))$ are monomorphisms we deduce that we have a short exact sequence $0 \longrightarrow \mathsf{H}\big(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)\big) \longrightarrow \mathsf{H}(G_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$. Since $\mathsf{H}(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big))$ is projective in $\mathsf{mod}\text{-}\mathcal{X}$, it follows that the map $\mathsf{H}(G_A) \longrightarrow \mathsf{H}(A)$ is a right Gorenstein-projective approximation of $\mathsf{H}(A)$. On the other hand let M be in $({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$ and let $K \longrightarrow M \longrightarrow X \longrightarrow K[1]$ be a triangle, where $K \in {}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}$ and $X \in \mathcal{X}$. Also let $M \longrightarrow A$ be a map. The composition $K \longrightarrow M \longrightarrow A \longrightarrow \Omega^n_{\mathcal{X}}(A[-1])[n+1]$, is zero since $K \in \mathcal{X}^{\top} = \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$ and $\Omega^n_{\mathcal{X}}(A[-1])[n+1] \in \mathcal{X}[n+1]$. This implies that the composition $K \longrightarrow M \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1]$ is zero and therefore the composition $M \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1]$ factorizes through the map $M \longrightarrow X$. Since $X \in \mathcal{X}$ and $\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1]$ lies in $\mathcal{X}[1]$, it follows that the composition $M \longrightarrow A \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1]$ is zero and therefore the map $M \longrightarrow A$ factorizes through the map $G_A \longrightarrow A$. This shows that $G_A \longrightarrow A$ is a right $\big(({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}\big)$-approximation of A, i.e. $({}^{\top}\mathcal{X} \cap \mathcal{X}^{\top}) \star \mathcal{X}$ is contravariantly finite in $\mathcal{T}$. $\square$

Corollary 8.6. *Let $\mathcal{X}$ be a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. Then the right adjoint of the inclusion $\underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X} \longrightarrow \underline{\mathsf{mod}}\text{-}\mathcal{X}$ is given by:*

$$\Omega^{-1}\Omega \ : \ \underline{\mathsf{mod}}\text{-}\mathcal{X} \longrightarrow \underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X}, \qquad \Omega^{-1}\Omega\mathsf{H}(A) = \mathsf{H}(G_A)$$

Let $A \in \mathcal{T}$ and consider the exact sequence $0 \longrightarrow \mathsf{H}\big(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)\big) \longrightarrow \mathsf{H}(G_A) \longrightarrow \mathsf{H}(A) \longrightarrow 0$ of the above Proposition, where $\mathsf{H}(G_A) \longrightarrow \mathsf{H}(A)$ is a right Gorenstein-projective approximation of $\mathsf{H}(A)$. Also consider the triangle, where the middle map is a left $\mathcal{X}$-approximation of G_A:

$$\mathsf{Cell}_{n-1}(G_A[n])[-n] \longrightarrow G_A \longrightarrow \Omega^n_{\mathcal{X}}(G_A[n]) \longrightarrow \mathsf{Cell}_{n-1}(G_A[n])[-n+1]$$

Forming the Octahedral diagram induced by the composition $\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big) \longrightarrow G_A \longrightarrow \Omega^n_{\mathcal{X}}(G_A[n])$ we obtain two triangles:

$$A \longrightarrow P^A \longrightarrow \mathsf{Cell}_{n-1}(A[-1])[2] \longrightarrow A[1] \tag{8.3}$$

$$\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big) \longrightarrow \Omega^n_{\mathcal{X}}(G_A[n]) \longrightarrow P_A \longrightarrow \Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)[1] \tag{8.4}$$

Finally set $\mathsf{H}(G^A) := \mathsf{Im}\,\big(\Omega^n_{\mathcal{X}}(G_A[n]) \longrightarrow \mathsf{Cell}_{n-1}(G_A[n])[-n+1]\big)$ for some object $G^A \in \mathcal{T}$.

Proposition 8.7. *Let $\mathcal{X}$ be a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, and let A be an object of $\mathcal{T}$.*

(i) $\mathsf{pd}\,\mathsf{H}(P_A) \leqslant 1$ *and the object $\mathsf{H}(G^A)$ is Gorenstein-projective.*

(ii) *There is a short exact sequence*

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(P^A) \longrightarrow \mathsf{H}(G^A) \longrightarrow 0$$

where the map $\mathsf{H}(A) \longrightarrow \mathsf{H}(P^A)$ is a left $(\mathsf{Proj}^{\leqslant 1}\mathsf{mod}\text{-}\mathcal{X})$-approximation of $\mathsf{H}(A)$.

Proof. Applying H to the triangle (8.2) we have an exact sequence $\mathsf{H}(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(G_A[n])) \longrightarrow \mathsf{H}(P_A) \longrightarrow 0$. However $\mathsf{H}(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(G_A[n]))$ is a monomorphism since by construction is the composition of the maps $\mathsf{H}(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)) \longrightarrow \mathsf{H}(G_A)$ and $\mathsf{H}(G_A) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(G_A[n])$ which are monomorphisms. Hence we have short exact sequence $0 \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(G_A[n])) \longrightarrow \mathsf{H}(P_A) \longrightarrow 0$ and therefore $\mathsf{pd}\,\mathsf{H}(P^A) \leqslant 1$ since the objects $\Omega^n_{\mathcal{X}}(G_A[n])$ and $\Omega^1_{\mathcal{X}}\big(\Omega^n_{\mathcal{X}}(A[-1])[n+1]\big)$ lie in $\mathcal{X}$. On the other hand applying the functor H to the triangle (8.3) and using that $\mathsf{H}(\mathsf{Cell}_{n-1}(G_A[n])[-n]) = 0$, since $\mathsf{Cell}_{n-1}(G_A[n])[-n] \in \mathcal{X}[-n] \star \cdots \star X[-1]$, we have an exact sequence $0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(P^A) \longrightarrow \mathsf{H}(G^A) \longrightarrow 0$ and $\mathsf{H}(G^A)$ is a subobject of $\mathsf{H}(\mathsf{Cell}_{n-1}(G_A[n])[-n+1])$. Since $\mathsf{Cell}_{n-1}(G_A[n])[-n+1]$ lies in $\mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1] \star \mathcal{X}$ and H kills the objects of $\mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, it follows that $\mathsf{H}(G^A)$ is a subobject of a projective object and therefore it is Gorenstein-projective. Clearly then the map $\mathsf{H}(A) \longrightarrow \mathsf{H}(P^A)$ is a left $\mathsf{Proj}^{\leqslant 1}\,\mathsf{mod}\text{-}\mathcal{X}$-approximation of $\mathsf{H}(A)$, since $\mathsf{Ext}^1(M, N) = 0$, for any Gorenstein-projective object M and any object N with finite projective dimension. $\square$

8.2. **Representation Dimension.** Recall that an additive category is called *Krull-Schmidt* if any of its objects is a finite coproduct of indecomposable objects and any indecomposable object has local endomorphism ring. A Krull-Schmidt category has *finite representation type* if there are finitely many indcomposable objects up to isomorphism. Now let $\mathscr{A}$ be an abelian category with enough projectives. We say that $\mathscr{A}$ is of *finite Cohen-Macaulay type*, *finite CM-type* for short, if $\mathsf{GProj}\,\mathscr{A}$ is a Krull-Schmidt category of finite represenattion type. In this section we show that if $\mathcal{X}$ is a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ and the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is of finite CM-type, then $\mathsf{mod}\text{-}\mathcal{X}$ is equivalent to the category of finitely presented modules over a coherent ring of representation dimension $\leqslant 3$ in the sense of Auslander [3].

Lemma 8.8. *Let $\mathscr{A}$ be a 1-Gorenstein abelian category. If $\mathsf{Proj}\,\mathscr{A}$ is covariantly finite and $\mathsf{Inj}\,\mathscr{A}$ is contravariantly finite in $\mathscr{A}$, then the full subcategory*

$$\mathscr{E} := \mathsf{GProj}\,\mathscr{A} \oplus \mathsf{GInj}\,\mathscr{A} = \mathsf{add}\,\big\{X \oplus Y \in \mathscr{A} \ \mid \ X \in \mathsf{GProj}\,\mathscr{A}, \ \ Y \in \mathsf{GInj}\,\mathscr{A}\big\}$$

is functorially finite in $\mathscr{A}$ and: $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathscr{E} \leqslant 3$.

Proof. Since $\mathscr{A}$ has enough projectives and enough injectives, the full subcategories $\mathsf{Proj}\,\mathscr{A}$ and $\mathsf{Inj}\,\mathscr{A}$ are functorially finite in $\mathscr{A}$. Then by [7, Theorem 4.16] the full subcategories $\mathsf{GProj}\,\mathscr{A}$ and $\mathsf{GInj}\,\mathscr{A}$ are functorially finite in $\mathscr{A}$. We use throughout that, by Remark 8.2, $\mathsf{GProj}\,\mathscr{A} = \Omega\mathscr{A}$ and $\mathsf{GInj}\,\mathscr{A} = \Sigma\mathscr{A}$.

Let F be in $\mathscr{A}$ and consider the following exact sequences in $\mathsf{mod}\text{-}\mathcal{X}$:

$$0 \longrightarrow F \longrightarrow I \longrightarrow \Sigma(F) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \Omega I \longrightarrow P \longrightarrow I \longrightarrow 0$$

where $F \longrightarrow I$ is a monomorphism into an injective object I and $P \longrightarrow I$ is an epimorphism from a projective object P. The above exact sequences induce the following exact sequence:

$$0 \longrightarrow \Omega I \longrightarrow \Omega\Sigma F \longrightarrow F \longrightarrow 0 \tag{8.5}$$

Since $\mathscr{A}$ is 1-Gorenstein, it follows that $\Omega(I)$ is projective. Hence applying $\mathscr{A}(G,-)$, where G is Gorenstein-projective, to (8.5), we have $\mathsf{Ext}^1_{\mathscr{A}}(G,\Omega I) = 0$ and therefore the map $\Omega\Sigma(F) \longrightarrow F \longrightarrow 0$ is a right $(\mathsf{GProj}\,\mathscr{A})$-approximation of F. Since $\mathsf{Inj}\,\mathscr{A}$ is contravariantly finite in $\mathscr{A}$, there is an exact sequence $0 \longrightarrow \Omega_{\mathbf{I}}(F) \longrightarrow J \longrightarrow F$, where the map $J \longrightarrow F$ is a right $(\mathsf{Inj}\,\mathscr{A})$-approximation of F. Then $\Sigma\Omega_{\mathbf{I}}(F) = \mathsf{Im}(J \longrightarrow F)$ lies in $\mathsf{GInj}\,\mathscr{A} = \Sigma(\mathscr{A})$ and we claim that the map $0 \longrightarrow \Sigma\Omega_{\mathbf{I}}(F) \longrightarrow F$ is a right $\mathsf{GInj}\,\mathscr{A}$-approximation of F. Indeed let Z be a Gorenstein-injective object and $Z \longrightarrow F$ be a map in $\mathscr{A}$; then by definition there exists an exact sequence $0 \longrightarrow Z' \longrightarrow J' \longrightarrow Z \longrightarrow 0$, where J' is injective and Z' is Gorenstein-injective. Then the composition $J' \longrightarrow Z \longrightarrow F$ factors through J and we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] \arrow[d] & J' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{\mathbf{I}}(F) \arrow[r] & J \arrow[r] & F &
\end{tikzcd}$$

It follows that there exists a unique map $Z \longrightarrow \Sigma\Omega_{\mathbf{I}}(F) = \mathsf{Im}(J \to F)$ and then by diagram chasing it is easy to see that $Z \longrightarrow F$ factors through $Z \longrightarrow \Sigma\Omega_{\mathbf{I}}(F)$, i.e. $0 \longrightarrow \Sigma\Omega_{\mathbf{I}}(F) \longrightarrow F$ is a right $\mathsf{GInj}\,\mathscr{A}$-approximation of F. Taking the pull-back of 8.5 along the map $\Sigma\Omega_{\mathbf{I}}(F) \longrightarrow F$ we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega I \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & \Sigma\Omega_{\mathbf{I}}(F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega I \arrow[r] & \Omega\Sigma F \arrow[r] & F \arrow[r] & 0
\end{tikzcd} \tag{8.6}$$

where the middle map $G \longrightarrow \Omega\Sigma F$ is a monomorphism and therefore G is Gorenstein-projective. The above pull-back diagram induces an exact sequence

$$0 \longrightarrow G \longrightarrow \Omega\Sigma F \oplus \Sigma\Omega_{\mathbf{I}}(F) \longrightarrow F \longrightarrow 0 \tag{8.7}$$

where clearly the objects G and $\Omega\Sigma F \oplus \Sigma\Omega_{\mathbf{I}}(F)$ lie in $\mathscr{E}$. Using the pull-back diagram (8.6) it is easy to see that the map $\Omega\Sigma F \oplus \Sigma\Omega_{\mathbf{I}}(F) \longrightarrow F$ is a right $\mathscr{E}$-approximation of F. Hence $\mathscr{E}$ is contravariantly finite in $\mathscr{A}$. A dual construction gives that $\mathscr{E}$ is covariantly finite in $\mathscr{A}$. In particular $\mathsf{mod}\text{-}\mathscr{E}$ is abelian with enough projectives and the restricted Yoneda embedding $\mathsf{Y} : \mathscr{A} \longrightarrow \mathsf{mod}\text{-}\mathscr{E}$, $\mathsf{Y}(F) = \mathsf{Hom}(-,F)|_{\mathscr{E}}$ induces an equivalence between $\mathscr{E}$ and $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathscr{E}$. Applying Y to (8.6) we have then a projective resolution

$$0 \longrightarrow \mathsf{Y}(G) \longrightarrow \mathsf{Y}(\Omega\Sigma F \oplus \Sigma\Omega_{\mathbf{I}}(F)) \longrightarrow \mathsf{Y}(F) \longrightarrow 0 \tag{8.8}$$

in $\mathsf{mod}\text{-}\mathscr{E}$ and therefore $\mathsf{pd}\,\mathsf{Y}(F) \leqslant 1$. Finally let M be an arbitrary object of $\mathsf{mod}\text{-}\mathscr{E}$ and choose a projective presentation $\mathsf{Y}(W_1) \longrightarrow \mathsf{Y}(W_0) \longrightarrow M \longrightarrow 0$ in $\mathsf{mod}\text{-}\mathscr{E}$, where the W_i lie in $\mathscr{E}$. If $F = \mathsf{Ker}(W_1 \to W_0)$, then using (8.8) we have a projective resolution

$$0 \longrightarrow \mathsf{Y}(W_3) \longrightarrow \mathsf{Y}(W_2) \longrightarrow \mathsf{Y}(W_1) \longrightarrow \mathsf{Y}(W_0) \longrightarrow M \longrightarrow 0$$

where $W_3 = \mathsf{Y}(G)$ for some Gorenstein-projective object G and $W_2 = \mathsf{Y}\big(\Omega\Sigma(F) \oplus \Sigma\Omega_{\mathbf{I}}(F)\big)$. Hence $\mathsf{pd}\,M \leqslant 3$ and therefore $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathscr{E} \leqslant 3$. $\square$

Recall that if $\mathscr{A}$ is an abelian category, then the *representation dimension* $\mathsf{rep.dim}\,\mathscr{A}$ of $\mathscr{A}$ in the sense Auslander [3] is defined as follows. An object T of $\mathscr{A}$ is called *generator*, resp. *cogenerator*, if any object of $\mathscr{A}$ is a factor, resp. subobject, of a direct summand of a finite direct sum of copies of T. We call an object T of $\mathscr{A}$ *right coherent* if the full subcategory $\mathsf{add}\,T$ has weak kernels; dually T is called *left coherent* if $\mathsf{add}\,T$ has weak cokernels. For instance T is right, resp. left, coherent, if $\mathsf{add}\,T$ is contravariantly, resp. covariantly, finite. Finally T is called coherent if T is left and right coherent object. Then

$$\mathsf{rep.dim}\,\mathscr{A} = \inf\big\{\,\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathscr{A}}(T) \;\mid\; T \;\text{ is a coherent generator-cogenerator of }\; \mathscr{A}\big\}$$

Theorem 8.9. *Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and $\mathcal{X}$ a strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. If $\mathsf{mod}\text{-}\mathcal{X}$ is of finite Cohen-Macauly type, then $\mathsf{rep.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 3$. More precisely there is a strong $(n+1)$-cluster tilting coherent object $T \in \mathcal{X}$ such that $\mathsf{mod}\text{-}\mathcal{X} \approx \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{T}}(T)$, the ring $\mathsf{End}_{\mathcal{T}}(T)$ is coherent and the representation dimension of $\mathsf{mod}\text{-}\mathsf{End}_{\mathcal{T}}(T)$ is at most 3.*

Proof. Since $\mathsf{mod}\text{-}\mathcal{X}$ is of finite Cohen-Macaulay type, clearly $\mathcal{X} = \mathsf{add}\,T$ for some $(n+1)$-cluster tilting object $T \in \mathcal{T}$ and then $\mathsf{mod}\text{-}\mathcal{X} \approx \mathsf{mod}\text{-}\mathsf{End}_{\mathcal{T}}(T)$. For the same reason $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X} = \mathsf{add}\,Z$ and $\mathsf{GInj}\,\mathsf{mod}\text{-}\mathcal{X} = \mathsf{add}\,W$, where $Z, W \in \mathsf{mod}\text{-}\mathcal{X}$. Then the object $G := Z \oplus W$ is a coherent object which is a generator-cogenerator of $\mathsf{mod}\text{-}\mathcal{X}$, and then Lemma 8.8 shows that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathsf{mod}\text{-}\mathcal{X}}(H) \leqslant 3$. It follows that $\mathsf{rep.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant 3$. □

Recall that a finite-dimensional k-algebra Λ over a field k is called *of finite CM-type* if the full subcategory $\mathsf{Gproj}\,\Lambda$ of finitely generated Gorenstein-projective Λ-modules is of finite representation type.

Corollary 8.10. *Let $\mathcal{T}$ be a triangulated category and T a strong $(n+1)$-cluster tilting object of $\mathcal{T}$. Assume that the cluster-tilted algebra $\mathsf{End}_{\mathcal{T}}(T)$ is of finite Cohen-Macaulay type. Then the algebra $\mathsf{End}_{\mathcal{T}}(T)$ has representation dimension at most 3.*

9. Certain Cluster Tilted Subcategories are Stably Calabi-Yau

We have seen that the category $\mathsf{mod}\text{-}\mathcal{X}$ of coherent functors over a $(n-k)$-strong $(n+1)$-cluster tilting subcategory $\mathcal{X}$ of a triangulated category $\mathcal{T}$ is k-Gorenstein, provided that $0 \leqslant k \leqslant 1$ or $k \leqslant \frac{n+1}{2}$, if $2 \leqslant k \leqslant n-1$. In this section we show that if the triangulated category $\mathcal{T}$ is $(n+1)$-Calabi-Yau, then the triangulated stable category modulo projectives of the Gorenstein-projective objects of $\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau in case $0 \leqslant k \leqslant 1$, and under an additional assumption if $2 \leqslant k \leqslant \frac{n+1}{2}$. This generalizes a basic result of Keller-Reiten [21] who treated the case $n = 1$.

Throughout let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a fixed $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geqslant 1$.

9.1. **Serre Functors.** Assume that the triangulated category $\mathcal{T}$ is k-linear with split idempotents over a field k, all Hom-spaces are finite-dimensional, and admits a Serre functor $\mathbb{S}$. So $\mathbb{S}\colon \mathcal{T} \longrightarrow \mathcal{T}$ is a triangulated equivalence and there are natural isomorphisms

$$\mathsf{D}\mathcal{T}(A,B) \xrightarrow{\cong} \mathcal{T}(B, \mathbb{S}(A)) \tag{$*$}$$

where D denotes duality with respect to the base field k.

For any object $A \in \mathcal{T}$ we consider triangles

$$X^1_A \longrightarrow X^0_A \longrightarrow \mathsf{Cell}_1(A) \longrightarrow X^1_A[1] \quad \text{ and } \quad X^A_1[-1] \longrightarrow \mathsf{Cell}^1(A) \longrightarrow X^A_0 \longrightarrow X^A_1 \tag{9.1}$$

where the maps $X^1_A \longrightarrow \Omega^1_{\mathcal{X}}(A)$ and $X^0_A \longrightarrow A$ are right $\mathcal{X}$-approximations and the maps $A \longrightarrow X^A_0$ and $\Sigma^1_{\mathcal{X}}(A) \longrightarrow X^A_1$ are left $\mathcal{X}$-approximations. Then

$$\mathsf{Cell}_1(A) \;\in\; \mathcal{X} \star \mathcal{X}[1] \;\text{ and }\; \mathsf{H}(\mathsf{Cell}_1(A)) \cong \mathsf{H}(A), \qquad \mathsf{Cell}^1(A) \;\in\; \mathcal{X}[-1] \star \mathcal{X} \;\text{ and }\; \mathsf{H}^{\mathsf{op}}(A) \cong \mathsf{H}(\mathsf{Cell}^1(A))$$

Recall that the transpose $\mathsf{Tr}(F)$, in the sense of Auslander-Bridger [4], of an object F in $\mathsf{mod}\text{-}\mathcal{X}$ is defined as follows. Let $\mathsf{H}(X^1) \longrightarrow \mathsf{H}(X^0) \longrightarrow F \longrightarrow 0$ be a projective presentation of F. Consider the duality functor $\mathsf{d}^r := \mathsf{Hom}(-, \mathsf{H}(?)|_{\mathcal{X}}) : \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$, defined by $\mathsf{d}^r(F) = \mathsf{Hom}(F, \mathsf{H}(?)|_{\mathcal{X}}) : \mathcal{X} \longrightarrow \mathscr{A}b$, where $\mathsf{Hom}(F, \mathsf{H}(?)|_{\mathcal{X}})(X) = \mathsf{Hom}(F, \mathsf{H}(X))$. Similarly the duality functor $\mathsf{d}^l : \mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ is defined and it is well-known that $(\mathsf{d}^r, \mathsf{d}^l) : \mathsf{mod}\text{-}\mathcal{X} \leftrightarrows \mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$ is an adjoint on the right pair of contravariant functors inducing a duality between $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}$ and $\mathsf{Proj}\,\mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$, a duality between $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$ and $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$, and finally a duality between $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ and $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$. Now the transpose $\mathsf{Tr}(F)$ of F is defined by $\mathsf{Tr}(F) = \mathsf{Coker}\big(\mathsf{d}^r\mathsf{H}(X^0) \longrightarrow \mathsf{d}^r\mathsf{H}(X^1)\big)$. Since $\mathsf{d}^r\mathsf{H} \cong \mathsf{H}^{\mathsf{op}}$, we have an exact sequence

$$0 \longrightarrow \mathsf{d}^r(F) \longrightarrow \mathsf{H}^{\mathsf{op}}(X^0) \longrightarrow \mathsf{H}^{\mathsf{op}}(X^1) \longrightarrow \mathsf{Tr}(F) \longrightarrow 0$$

In this way one obtains a functor $\mathsf{Tr} : \underline{\mathsf{mod}}\text{-}\mathcal{X} \longrightarrow \underline{\mathsf{mod}}\text{-}\mathcal{X}^{\mathsf{op}}$ which is well-known to be a duality. Now the duality D with respect to the base field k acts on $\mathsf{mod}\text{-}\mathcal{X}$ by $\mathsf{D}(F)(X) = \mathsf{D}(F(X))$.

Lemma 9.1. *There are isomorphisms:*

$$\mathsf{DTrH}(A) \cong \mathsf{H}(\mathbb{S}\mathsf{Cell}_1(A)[-1]), \quad \mathsf{TrDH}(A) \cong \mathsf{H}(\mathbb{S}^{-1}\mathsf{Cell}^1(A)[1]) \tag{9.2}$$

$$\mathsf{DTrH}^{\mathsf{op}}(A) \cong \mathsf{H}(\mathbb{S}^{-1}\mathsf{Cell}_1(A)[1]), \quad \mathsf{TrDH}^{\mathsf{op}}(A) \cong \mathsf{H}(\mathbb{S}\mathsf{Cell}^1(A)[-1]) \tag{9.3}$$

Proof. For simplicity we set $A^* = \mathsf{Cell}_1(A)$ and $A_* = \mathsf{Cell}^1(A)$. Applying H and H^{op} to the triangles in (9.1) we have exact sequences $\mathsf{H}(X^1_A) \longrightarrow \mathsf{H}(X^0_A) \longrightarrow \mathsf{H}(A^*) \longrightarrow 0$ and $\mathsf{H}^{\mathsf{op}}(X^A_1) \longrightarrow \mathsf{H}^{\mathsf{op}}(X^A_0) \longrightarrow \mathsf{H}^{\mathsf{op}}(A_*) \longrightarrow 0$. Applying the dual functors $\mathsf{Hom}(-, \mathsf{H}^{\mathsf{op}}(?)|_{\mathcal{X}}) : \mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}$ and $\mathsf{Hom}(-, \mathsf{H}(?)|_{\mathcal{X}}) : \mathsf{mod}\text{-}\mathcal{X} \longrightarrow \mathsf{mod}\text{-}\mathcal{X}^{\mathsf{op}}$ we then have exact sequences $\mathsf{H}(X^A_0) \longrightarrow \mathsf{H}(X^A_1) \longrightarrow \mathsf{TrH}^{\mathsf{op}}(A^*) \longrightarrow 0$ and $\mathsf{H}^{\mathsf{op}}(X^0_A) \longrightarrow \mathsf{H}^{\mathsf{op}}(X^1_A) \longrightarrow \mathsf{TrH}(A_*) \longrightarrow 0$. Finally applying the duality functor D and using $(*)$ we have exact sequences

$$0 \longrightarrow \mathsf{DTrH}(A^*) \longrightarrow \mathsf{H}(\mathbb{S}X^1_A) \longrightarrow \mathsf{H}(\mathbb{S}X^0_A) \quad \text{and} \quad 0 \longrightarrow \mathsf{DTrH}^{\mathsf{op}}(A_*) \longrightarrow \mathsf{H}^{\mathsf{op}}(\mathbb{S}^{-1}X^A_1) \longrightarrow \mathsf{H}^{\mathsf{op}}(\mathbb{S}^{-1}X^A_0)$$

However since $\mathbb{S}$ and $\mathbb{S}^{-1}$ are triangulated, we have triangles

$$\mathbb{S}(X^0_A[-1]) \longrightarrow \mathbb{S}(A^*[-1]) \longrightarrow \mathbb{S}X^1_A \longrightarrow \mathbb{S}X^0_A, \qquad \mathbb{S}^{-1}X^A_0 \longrightarrow \mathbb{S}^{-1}X^A_1 \longrightarrow \mathbb{S}^{-1}(A_*[1]) \longrightarrow \mathbb{S}^{-1}(X^A_0[1])$$

Since $\mathsf{H}(\mathbb{S}(X^0_A[-1])) = \mathcal{T}((\mathcal{X}, \mathbb{S}(X^0_A[-1])) = \mathsf{D}\mathcal{T}(X^0_A[-1], \mathcal{X}) = \mathsf{D}\mathcal{T}(X^0_A, \mathcal{X}[1]) = 0$ and $\mathsf{H}^{\mathsf{op}}(\mathbb{S}^{-1}(X^A_0[1])) = \mathcal{T}(\mathbb{S}^{-1}(X^A_0[1]), \mathcal{X}) = \mathcal{T}(X^A_0[1], \mathbb{S}\mathcal{X}) = \mathsf{D}\mathcal{T}(\mathcal{X}, X^A_0[1]) = 0$, we infer that

$$\mathsf{DTrH}(A^*) \xrightarrow{\cong} \mathsf{H}\mathbb{S}(A^*[-1]) \quad \text{and} \quad \mathsf{DTrH}^{\mathsf{op}}(A_*) \xrightarrow{\cong} \mathsf{H}^{\mathsf{op}}\mathbb{S}^{-1}(A_*[1])$$

Since $\mathsf{H}(A) \cong \mathsf{H}(A^*)$ and $\mathsf{H}^{\mathsf{op}}(A) \cong \mathsf{H}^{\mathsf{op}}(A_*)$, these reduce the isomorphisms:

$$\mathsf{DTrH}(A) \xrightarrow{\cong} \mathsf{H}\mathbb{S}(A^*[-1]) \quad \text{and} \quad \mathsf{DTrH}^{\mathsf{op}}(A) \xrightarrow{\cong} \mathsf{H}^{\mathsf{op}}\mathbb{S}^{-1}(A_*[1])$$

Similarly we get isomorphisms: $\mathsf{DTrH}^{\mathsf{op}}(A) \cong \mathsf{H}(\mathbb{S}^{-1}A^*[1])$ and $\mathsf{TrDH}^{\mathsf{op}}(A) \cong \mathsf{H}(\mathbb{S}A_*[-1])$. □

To proceed further we need the following.

Lemma 9.2. *The cluster tilted category* $\mathsf{mod}\text{-}\mathcal{X}$ *has Auslander-Reiten sequences. In particular Auslander-Reiten formula*

$$\mathsf{D}\underline{\mathsf{Hom}}\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\cong} \mathsf{Ext}^1\big(\mathsf{H}(B), \mathsf{DTr}(\mathsf{H}(A)\big) \tag{9.4}$$

holds for any objects $\mathsf{H}(A)$ *and* $\mathsf{H}(B)$ *in* $\mathsf{mod}\text{-}\mathcal{X}$. *Moreover there are isomorphisms:*

$$\overline{\mathsf{Hom}}\big(\mathsf{H}(B), \mathsf{H}(\mathbb{S}\mathsf{Cell}_1(A)[-1])\big) \xleftarrow{\cong} \mathsf{DExt}^1\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\cong} \underline{\mathsf{Hom}}\,\big(\mathsf{H}(\mathbb{S}^{-1}\mathsf{Cell}^1(B)[1]), \mathsf{H}(A)\big)$$

Proof. Since $\mathcal{X}$ is an $(n+1)$-cluster tilting subcategory, it follows that $\mathcal{X}$ is functorially finite. On the other hand by using $(*)$, we have isomorphisms, $\forall X \in \mathcal{X}$:

$$\mathsf{DH}(X) \xrightarrow{\cong} \mathsf{D}\mathcal{T}(\mathcal{X}, X) \xrightarrow{\cong} \mathcal{T}(X, \mathbb{S}\mathcal{X}) \xrightarrow{\cong} \mathcal{T}(\mathbb{S}^{-1}X, \mathcal{X}) \xrightarrow{\cong} \mathsf{H}^{\mathsf{op}}(\mathbb{S}^{-1}X),$$

$$\mathsf{DH}^{\mathsf{op}}(X) \xrightarrow{\cong} \mathsf{D}\mathcal{T}(X, \mathcal{X}) \xrightarrow{\cong} \mathcal{T}(\mathcal{X}, \mathbb{S}X) \xrightarrow{\cong} \mathsf{H}(\mathbb{S}X)$$

which show that k-duals of contravariant or covariant representable functors over $\mathcal{X}$ are coherent. This clearly implies, by [5], that $\mathsf{mod}\text{-}\mathcal{X}$ is a dualizing k-variety. In particular has Auslander-Reiten sequences and Auslander-Reiten formula (9.4) holds. The remaining isomorphisms follow by using Lemma 9.1. □

Finally we shall need the following observation.

Lemma 9.3. *The Serre functor* $\mathbb{S} : \mathcal{T} \longrightarrow \mathcal{T}$ *induces an equivalence*

$$\mathbb{S} : \mathcal{X} \xrightarrow{\approx} \mathcal{X}[n+1]$$

Proof. Consider the triangle

$$\Omega^n_{\mathcal{X}}(\mathbb{S}(X[-1])[n-1] \longrightarrow \mathsf{Cell}_{n-1}(\mathbb{S}(X[-1])) \longrightarrow \mathbb{S}(X[-1]) \longrightarrow \Omega^n_{\mathcal{X}}(\mathbb{S}(X[-1])[n]$$

Since the functor $\mathbb{S}$ is triangulated, we have a triangle

$$\Omega^n_{\mathcal{X}}(\mathbb{S}(X[-1])[n] \longrightarrow \mathsf{Cell}_{n-1}(\mathbb{S}(X[-1]))[1] \longrightarrow \mathbb{S}(X) \longrightarrow \Omega^n_{\mathcal{X}}(\mathbb{S}(X[-1])[n+1]$$

By Serre duality the middle map corresponds to an element of $\mathsf{D}\mathcal{T}(X, \mathsf{Cell}_{n-1}(\mathbb{S}(X[-1]))[1])$. The last space is zero since $\mathsf{Cell}_{n-1}(\mathbb{S}(X[-1]))[1] \in \mathcal{X}[1] \star \mathcal{X}[2] \star \cdots \star \mathcal{X}[n]$, see Proposition 4.2. Hence the middle map of the triangle above is zero and therefore $\mathbb{S}(X)$ lies in $\mathcal{X}[n+1]$ as a direct summand of $\Omega^n_{\mathcal{X}}(\mathbb{S}(X[-1])[n+1]$. Therefore $\mathbb{S}(\mathcal{X}) \subseteq \mathcal{X}[n+1]$. Dually if $X[n+1] \in \mathcal{X}[n+1]$, then for any i with $1 \leqslant i \leqslant n$, we have: $\mathcal{T}(\mathbb{S}^{-1}(X[n+1]), \mathcal{X}[i]) = \mathcal{T}(X[n+1], \mathbb{S}(\mathcal{X}[i])) \cong \mathsf{D}\mathcal{T}(\mathcal{X}[i], X[n+1]) = 0$. Hence $\mathbb{S}^{-1}(X[n+1]) \in {}^{\top}_{n}\mathcal{X} = \mathcal{X}$ and therefore $\mathbb{S}^{-1}(\mathcal{X}[n+1]) \subseteq \mathcal{X}$ and the assertion follows. □

9.2. $(n+1)$**-Calabi-Yau Categories.** From now on we assume that: $\mathcal{T}$ is $(n+1)$-*Calabi-Yau*, i.e. there is an isomorphism of triangulated functors:

$$\mathbb{S}(?) \xrightarrow{\cong} (?)[n+1]$$

Then we have natural isomorphisms

$$\mathsf{D}\,\mathsf{Hom}_{\mathcal{T}}(A,B) \xrightarrow{\cong} \mathsf{Hom}_{\mathcal{T}}(B,A[n+1]) \tag{9.5}$$

Lemma 9.4. *Let A be an object in $\mathcal{X}\star\mathcal{X}[1]$. Then we have the following*

(i) $\mathsf{DTrH}(A) \cong \mathsf{H}(A[n])$.

(ii) *If $n\geqslant 2$, then:* $\mathsf{H}(A[t])=0$, $1\leqslant t\leqslant n-1$.

(iii) *For any $\mathsf{H}(B)\in\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$, there is a natural isomorphism:*

$$\mathsf{D}\underline{\mathsf{Hom}}\big(\mathsf{H}(A),\mathsf{H}(B)\big) \xrightarrow{\cong} \underline{\mathsf{Hom}}\big(\Omega^1\mathsf{H}(B),\mathsf{H}(A[n])\big)$$

Proof. (i) Since $A\in\mathcal{X}\star\mathcal{X}[1]$, we may take $A=\mathsf{Cell}_1(A)$ and then by Lemma 9.1, we have an isomorphism $\mathsf{DTr}(\mathsf{H}(A))\cong\mathsf{H}(A[n])$.

(ii) Since $n\geqslant 2$, and $A\in\mathcal{X}\star\mathcal{X}[1]$, we have $A[t]\in\mathcal{X}[t]\star\mathcal{X}[t+1]$ which is contained in $\mathcal{X}^{\top}=\mathcal{X}[1]\star\cdots\star\mathcal{X}[n]$, for $1\leqslant t\leqslant n-1$. It follows that $\mathsf{H}(A[t])=0$, $1\leqslant t\leqslant n-1$.

(iii) By Lemma 9.2 we have an isomorphism $\mathsf{D}\underline{\mathsf{Hom}}\big(\mathsf{H}(A),\mathsf{H}(B)\big)\cong\mathsf{Ext}^1\big(\mathsf{H}(B),\mathsf{DTr}(\mathsf{H}(A)\big)$. Hence (i) gives us an isomorphism $\mathsf{D}\underline{\mathsf{Hom}}\big(\mathsf{H}(A),\mathsf{H}(B)\big)\cong\mathsf{Ext}^1\big(\mathsf{H}(B),\mathsf{H}(A[n])\big)$. Since $\mathsf{H}(B)$ is Gorenstein-projective, by Remark 8.2 we have an isomorphism $\mathsf{Ext}^1\big(\mathsf{H}(B),\mathsf{H}(A[n])\big)\cong\underline{\mathsf{Hom}}\,\big(\Omega^1\mathsf{H}(B),\mathsf{H}(A[n])\big)$. $\square$

To proceed we need the following two preliminary results.

Lemma 9.5. *Let $n\geqslant 2$, and A be an object in $\mathcal{X}\star\mathcal{X}[1]$ such that $\mathsf{H}(A)$ is Gorenstein-projective. Let*

$$C \longrightarrow A \longrightarrow B \longrightarrow C[1]$$

be a triangle in $\mathcal{T}$. If C lies in $\mathcal{X}[-n]\star\mathcal{X}[-n+1]\star\cdots\star\mathcal{X}[-1]$, then there exists a short exact sequence

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B) \longrightarrow \mathsf{H}(C[1]) \longrightarrow 0$$

which remains exact after the application of the functor $\mathsf{Hom}(-,\mathsf{H}(X))$, $\forall X\in\mathcal{X}$. Moreover if $\mathsf{H}(C[1])$ is Gorenstein-projective, then so is $\mathsf{H}(B)$. The converse holds if $\mathsf{mod}\text{-}\mathcal{X}$ is Gorenstein.

Proof. Since $\mathsf{H}(A)$ is Gorenstein-projective, there is a monomorphism $\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$, where $X\in\mathcal{X}$. Consider the composition $\mathsf{H}(C)\longrightarrow\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$. Since A lies in $\mathcal{X}\star\mathcal{X}[1]$, the monomorphism $\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$ is induced by a map $A\longrightarrow X$. The composition $C\longrightarrow A\longrightarrow X$ is clearly zero, since C lies in $\mathcal{X}[-n]\star\mathcal{X}[-n+1]\star\cdots\star\mathcal{X}[-1]$. Therefore the map $A\longrightarrow X$ factorizes through $\mathsf{H}(B)$. As a consequence the composition $\mathsf{H}(C)\longrightarrow\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$ is zero. Since $\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$ is a monomorphism and $\mathsf{H}(A[1])=0$, since $A\in\mathcal{X}\star\mathcal{X}[1]$, we infer that the map $\mathsf{H}(C)\longrightarrow\mathsf{H}(A)$ is zero and therefore we have an exact sequence:

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(B) \longrightarrow \mathsf{H}(C[1]) \longrightarrow 0 \tag{9.6}$$

and in addition, as the above argument shows, any map $\mathsf{H}(A)\longrightarrow\mathsf{H}(X)$, $X\in\mathcal{X}$, factorizes through $\mathsf{H}(B)$. It follows that the exact sequence (9.6) remains exact after the application of the functor $\mathsf{H}(X)$, $\forall X\in\mathcal{X}$. If $\mathsf{H}(C[1])$ is Gorenstein-projective, then so is $\mathsf{H}(B)$, since $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$ is well-known to be closed under extensions. Conversely let $\mathsf{mod}\text{-}\mathcal{X}$ be Gorenstein, say k-Gorenstein, and let $\mathsf{H}(B)$ be Gorenstein-projective. It is well-known, see [7], that $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$ consists of all objects F such that $\mathsf{Ext}^k(F,\mathsf{H}(X))=0$, $\forall X\in\mathcal{X}$, $\forall k\geqslant 1$. Applying $\mathsf{Hom}(-,\mathsf{H}(X))$, $\forall X\in\mathcal{X}$, to the exact sequence (9.6) we have trivially that $\mathsf{Ext}^k(\mathsf{H}(C[1]),\mathsf{H}(X))=0$, $\forall X\in\mathcal{X}$, $\forall k\geqslant 1$, so $\mathsf{H}(C[1])$ is Gorenstein-projective. $\square$

Lemma 9.6. *Let $\mathcal{T}$ be an $(n+1)$-Calabi-Yau triangulated category over a field k, $n\geqslant 1$, and let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ such that the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein, $k\geqslant 0$. Assume that for any $A\in\mathcal{X}\star\mathcal{X}[1]$ such that $\mathsf{H}(A)$ is Gorenstein-projective, there is a natural isomorphism:*

$$\Omega^{-(n+1)}\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-k}\Omega^k\mathsf{H}(A[n])$$

in $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$. Then $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.

Proof. Since $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein, it follows from Remark 8.2 that $\Omega^k\mathsf{H}(C)\in\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$, $\forall C\in\mathcal{T}$, and moreover the functor $\Omega^{-k}\Omega^k:\underline{\mathsf{mod}}\text{-}\mathcal{X}\longrightarrow\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is a right adjoint of the inclusion $\underline{\mathsf{mod}}\text{-}\mathcal{X}\longrightarrow\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$. Then by Lemma 9.4, we have natural isomorphisms

$$\mathsf{D}\underline{\mathsf{Hom}}\big(\mathsf{H}(A),\mathsf{H}(B)\big) \xrightarrow{\cong} \underline{\mathsf{Hom}}\big(\Omega^1\mathsf{H}(B),\mathsf{H}(A[n])\big) \xrightarrow{\cong} \underline{\mathsf{Hom}}\big(\Omega^1\mathsf{H}(B),\Omega^{-k}\Omega^k\mathsf{H}(A[n])\big) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \underline{\mathsf{Hom}}\big(\Omega^1\mathsf{H}(B), \Omega^{-(n+1)}\mathsf{H}(A)\big) \xrightarrow{\cong} \underline{\mathsf{Hom}}\big(\mathsf{H}(B), \Omega^{-(n+2)}\mathsf{H}(A)\big)$$

for any Gorenstein-projective objects $\mathsf{H}(B)$ and $\mathsf{H}(A)$, with $A \in \mathcal{X}\star\mathcal{X}[1]$. Since any object $F = \mathsf{H}(C)$ of $\mathsf{mod}\text{-}\mathcal{X}$ is isomorphic to an object of the form $\mathsf{H}(A)$, where $A \in \mathcal{X} \star \mathcal{X}[1]$, namely $A = \mathsf{Cell}_1(C)$, the above natural isomorphism holds for all Gorenstein-projective objects. Hence the functor $\Omega^{-(n+2)} : \underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X} \longrightarrow \underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ serves as a Serre functor in $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$, i.e. $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau. □

Now we are ready to state and prove the first main result of this section. Note that the case $n = 1$ is due to Keller-Reiten [21].

Theorem 9.7. *Let $\mathcal{T}$ be a k-linear triangulated category over a field k with finite-dimensional Hom-spaces. Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ and assume that $\mathcal{X}$ is $(n-k)$-strong, $0 \leqslant k \leqslant 1$, $n \geqslant 1$. If $\mathcal{T}$ is $(n+1)$-Calabi-Yau, then the stable triangulated category $\underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.*

We split the proof in two steps $k = 0$ and $k = 1$. In each step we need to treat separately the cases $n = 1$ and $n \geqslant 2$. Note that under the imposed assumptions, the category $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein. More precisely for $k = 0$, $\mathsf{mod}\text{-}\mathcal{X}$ is Frobenius, so $\mathsf{mod}\text{-}\mathcal{X} = \mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$ and $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X} = \underline{\mathsf{mod}}\text{-}\mathcal{X}$. If $k = 1$ then the category $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein and $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X} = \Omega\,\mathsf{mod}\text{-}\mathcal{X}$ coincides with the full subcategory of $\mathsf{mod}\text{-}\mathcal{X}$ consisting of the subobjects of the projective objects.

Lemma 9.8. *Let $\mathcal{X}$ be an $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geqslant 1$. Assume that $\mathcal{X}$ is $(n-1)$-strong, if $n \geqslant 2$. Then $\forall A \in \mathcal{X} \star \mathcal{X}[1]$, such that $\mathsf{H}(A)$ is Gorenstein-projective, there is an isomorphism in $\underline{\mathsf{mod}}\text{-}\mathcal{X}$:*

$$\mathsf{H}(A) \xrightarrow{\cong} \Omega^{n+1}\mathsf{H}(A[n])$$

Proof. **1.** Assume that $n = 1$. Then $\mathcal{T} = \mathcal{X}\star\mathcal{X}[1]$, and $\mathcal{X}$ is a 2-cluster tilting subcategory of the 2-Calabi-Yau category $\mathcal{T}$. By Lemma 8.4, there exists a triangle $X^1_{A[1]}[-1] \longrightarrow X^0_{A[1]}[-1] \longrightarrow A \longrightarrow X^1_{A[1]}$, where the last map is a left $\mathcal{X}$-approximation of A, $X^i_{A[1]}$ lie in $\mathcal{X}$, and the map $X^0_{A[1]}[-1] \longrightarrow A$ is $\mathcal{X}$-ghost. Hence applying H to the above triangle and using that $\mathcal{T}(\mathcal{X}, \mathcal{X}[1]) = 0$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(X^1_{A[1]}) \longrightarrow \mathsf{H}(X^0_{A[1]}) \longrightarrow \mathsf{H}(A[1]) \longrightarrow 0$$

Hence in $\underline{\mathsf{mod}}\text{-}\mathcal{X}$ we have an isomorphism: $\mathsf{H}(A) \cong \Omega^2\mathsf{H}(A[1])$ as required.

2. Assume that $n \geqslant 2$ and $\mathcal{X}$ is $(n-1)$-strong. Consider the triangles

$$\Omega^k_{\mathcal{X}}(A[n]) \longrightarrow X^{k-1}_{A[n]} \longrightarrow \Omega^{k-1}_{\mathcal{X}}(A[n]) \longrightarrow \Omega^k_{\mathcal{X}}(A[n])[1] \qquad (T^k_{A[n]})$$

Since by Lemma 9.4, $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leqslant i \leqslant n-1$, applying H to the triangle $(T^1_{A[n]})$, we have exact sequences

$$0 \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^0_{A[n]}) \longrightarrow \mathsf{H}(A[n]) \longrightarrow 0 \qquad (9.7)$$

$$\mathsf{H}(X^0_{A[n]}[-n]) \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])[-n+1]) \longrightarrow 0 \qquad (9.8)$$

and isomorphisms:

$$\mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])[-k]) = 0, \quad 1 \leqslant k \leqslant n-2 \qquad (9.9)$$

By Lemma 9.5, the map $\mathsf{H}(X^0_{A[n]}[-n]) \longrightarrow \mathsf{H}(A)$ is zero and therefore (9.8) induces an isomorphism:

$$\mathsf{H}(A) \xrightarrow{\cong} \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])[-n+1]) \qquad (9.10)$$

Using (9.9) and applying H to the triangle $(T^2_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^1_{A[n]}) \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])) \longrightarrow 0$$

and isomorphisms:

$$\mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])[-k]) = 0, \quad 1 \leqslant k \leqslant n-3$$

$$\mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])[-n+1]) \xrightarrow{\cong} \mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])[-n+2]) \qquad (9.11)$$

Similarly applying H to the triangle $(T^3_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^3_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^2_{A[n]}) \longrightarrow \mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])) \longrightarrow 0$$

and isomorphisms:

$$\mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])[-k]) = 0, \quad 1 \leqslant k \leqslant n-4$$

$$\mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])[-n+k]) \xrightarrow{\cong} \mathsf{H}(\Omega^3_{\mathcal{X}}(A[n])[-n+k+1]), \;\; k = 1, 2 \qquad (9.12)$$

Continuing inductively in this way, and applying H to the triangle $(T^{n-1}_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{n-2}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

and isomorphisms:

$$\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) = 0$$

$$\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-k]) \xrightarrow{\cong} \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-k+1]), \quad 2 \leqslant k \leqslant n-1 \tag{9.13}$$

From (9.10) - (9.13), it follows that we have isomorphosms:

$$\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \xrightarrow{\cong} \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-2]) \xrightarrow{\cong} \mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])[-3]) \xrightarrow{\cong} \cdots$$

$$\cdots \xrightarrow{\cong} \mathsf{H}(\Omega^{2}_{\mathcal{X}}(A[n])[-n+2]) \xrightarrow{\cong} \mathsf{H}(\Omega^{1}_{\mathcal{X}}(A[n])[-n+1]) \xrightarrow{\cong} \mathsf{H}(A) \tag{9.14}$$

On the other hand the short exact sequences $0 \longrightarrow \mathsf{H}(\Omega^k_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{k-1}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])) \longrightarrow 0$, $1 \leqslant k \leqslant n-1$, shows that in $\underline{\mathsf{mod}}$-$\mathcal{X}$ we have isomorphisms:

$$\Omega^k\mathsf{H}(A[n]) \xrightarrow{\cong} \mathsf{H}(\Omega^k_{\mathcal{X}}(A[n])), \quad 0 \leqslant k \leqslant n-1 \tag{9.15}$$

Finally applying H to the triangle $(T^n_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^{n}_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{n-1}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

Since $\Omega^n_{\mathcal{X}}(A[n]) := X^n_{A[n]}$ lies in $\mathcal{X}$, it follows that $\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \cong \Omega^2\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n]))$. Hence we have an isomorphism in $\underline{\mathsf{mod}}$-$\mathcal{X}$:

$$\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \xrightarrow{\cong} \Omega^2\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{n+1}\mathsf{H}(A[n])$$

Putting things together we have an isomorphism $\mathsf{H}(A) \cong \Omega^{n+1}\mathsf{H}(A[n])$ in $\underline{\mathsf{mod}}$-$\mathcal{X}$, as required. □

Proposition 9.9. *Let $\mathcal{X}$ be an n-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$. If $\mathcal{T}$ is $(n+1)$-Calabi-Yau, then $\underline{\mathsf{mod}}$-$\mathcal{X}$ is $(n+2)$-Calabi-Yau.*

Proof. Since $\mathcal{X}$ is n-strong, we have $\mathcal{X} = \mathcal{X}[n+1]$ and therefore mod-$\mathcal{X}$ is Frobenius, so all objects of mod-$\mathcal{X}$ are Gorenstein-projective. It follows that $\underline{\mathsf{mod}}$-$\mathcal{X}$ is triangulated.

1. First let $n = 1$. Then $\mathcal{X} = \mathcal{X}[2]$ and $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1]$. By Lemma 9.8 we have an isomorphism $\Omega^2\mathsf{H}(A[1]) \cong \mathsf{H}(A)$. Since any object of mod-$\mathcal{X}$ is Gorenstein-projective, this is equivalent to $\Omega^{-2}\mathsf{H}(A) \cong \mathsf{H}(A[1])$. Then by Lemma 9.6 we infer that $\underline{\mathsf{mod}}$-$\mathcal{X}$ is 3-Calabi-Yau.

2. Now let $n \geqslant 2$. Let A be an object in $\mathcal{X} \star \mathcal{X}[1]$. Then by Lemma 9.8, we have an isomorphism $\mathsf{H}(A) \cong \Omega^{n+1}\mathsf{H}(A[n])$ or equivalently since $\underline{\mathsf{mod}}$-$\mathcal{X}$ is triangulated, $\Omega^{-(n+1)}\mathsf{H}(A) \cong \mathsf{H}(A[n])$. Then by Lemma 9.6 we infer that $\underline{\mathsf{mod}}$-$\mathcal{X}$ is $(n+2)$-Calabi-Yau. □

Now we treat the case $k = 1$. Note that the following result was proved independently by Iyama-Oppermann, see [17].

Proposition 9.10. *Let $\mathcal{X}$ be an $(n-1)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $n \geqslant 1$. If $\mathcal{T}$ is $(n+1)$-Calabi-Yau, then the stable triangulated category $\underline{\mathsf{GProj}}\,\mathsf{mod}$-$\mathcal{X}$ is $(n+2)$-Calabi-Yau.*

Proof. We treat separately the cases $n = 1$, where the $(n-1)$-strong condition is vacuous, and $n \geqslant 2$.

1. Assume that $n = 1$. We have that $\mathcal{X}$ is a 2-cluster tilting subcategory of the 2-Calabi-Yau category $\mathcal{T}$ and $\mathcal{T} = \mathcal{X} \star \mathcal{X}[1]$. Let A be in $\mathcal{T}$ such that $\mathsf{H}(A)$ is Gorenstein-projective. Then by Lemma 9.8 we know that $\Omega^2\mathsf{H}(A[1]) \cong \mathsf{H}(A)$. Since mod-$\mathcal{X}$ is 1-Gorenstein, it follows that $\Omega\mathsf{H}(A[1])$ is Gorenstein-projective. Then applying Ω^{-1} to the last isomorphism we obtain $\Omega\mathsf{H}(A[1]) \cong \Omega^{-1}\mathsf{H}(A)$ and therefore $\Omega^{-1}\Omega\mathsf{H}(A[1]) \cong \Omega^{-2}\mathsf{H}(A)$. Then by Lemma 9.6 it follows that $\underline{\mathsf{GProj}}\,\mathsf{mod}$-$\mathcal{X}$ is 3-Calabi-Yau.

2. Assume that $n \geqslant 2$. By Lemma 9.8, for any object $A \in \mathcal{X}\star\mathcal{X}[1]$, such that $\mathsf{H}(A)$ is Gorenstein-projective, there is a natural isomorphism in $\underline{\mathsf{mod}}$-$\mathcal{X}$:

$$\Omega^{n+1}\mathsf{H}(A[n]) \xrightarrow{\cong} \mathsf{H}(A)$$

Since $\Omega\mathsf{H}(A[n])$ is Gorenstein-projective, the above isomorphism gives:

$$\Omega^{-n}\mathsf{H}(A) \xrightarrow{\cong} \Omega^1\mathsf{H}(A[n]), \quad \text{hence} \quad \Omega^{-n-1}\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-1}\Omega^1\mathsf{H}(A[n])$$

It follows by Lemma 9.6 that $\underline{\mathsf{GProj}}\,\mathsf{mod}$-$\mathcal{X}$ is $(n+2)$-Calabi-Yau. □

9.3. **Higher Gorenstein categories.** We have seen that if $\mathcal{X}$ is an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$ and $n \geqslant 2k-1$, then the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein. Moreover if $0 \leqslant k \leqslant 1$ and if $\mathcal{T}$ is $(n+1)$-Calabi-Yau, then the stable triangulated category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.

In this subsection we prove that an analogous statement holds for $2 \leqslant k \leqslant n-1$ under an additional assumption.

Theorem 9.11. *Let $\mathcal{T}$ be an $(n+1)$-Calabi-Yau triangulated category over a field k. Let $\mathcal{X}$ be an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, where $n \geqslant 2k-1$, $0 \leqslant k \leqslant n-1$. Assume that any object $\mathsf{H}(C)$, where $C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, has finite projective or injective dimension.*

Then the stable triangulated category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.

Proof. First note that since by Theorem 7.5, the cluster tilted category $\mathsf{mod}\text{-}\mathcal{X}$ is k-Gorenstein, then by Remark 8.2, we have that $\Omega^k\mathsf{H}(A)$ lies in $\mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$, $\forall A \in \mathcal{T}$. We use throughout that if an object of $\mathsf{mod}\text{-}\mathcal{X}$ has finite projective dimension, then its projective dimension is at most k.

We shall show first that for any $A \in \mathcal{X} \star \mathcal{X}[1]$ such that $\mathsf{H}(A)$ is Gorenstein-projective, there is a natural isomorphism in the stable category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$:

$$\Omega^{-(n+1)}\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-k}\Omega^k\mathsf{H}(A[n])$$

For $1 \leqslant t \leqslant n$, we have triangles

$$\Omega^t_{\mathcal{X}}(A[n]) \longrightarrow X^{t-1}_{A[n]} \longrightarrow \Omega^{t-1}_{\mathcal{X}}(A[n]) \longrightarrow \Omega^t_{\mathcal{X}}(A[n])[1] \qquad (T^t_{A[n]})$$

where $\Omega^t_{\mathcal{X}}(A[n]) \in \mathcal{X}$ and $\Omega^0_{\mathcal{X}}(A[n]) = A[n]$. Using that, by Lemma 9.4, $\mathcal{T}(\mathcal{X}, A[i]) = 0$, $1 \leqslant i \leqslant n-1$, and $\mathcal{T}(\mathcal{X}, \mathcal{X}[-i]) = 0$, $1 \leqslant i \leqslant n-k$, we deduce a short exact sequence:

$$0 \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^0_{A[n]}) \longrightarrow \mathsf{H}(A[n]) \longrightarrow 0$$

and isomorphisms:

$$\mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])[-t]) = 0, \quad 1 \leqslant t \leqslant n-k$$

Using this and applying H to the triangle $(T^2_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^1_{A[n]}) \longrightarrow \mathsf{H}(\Omega^1_{\mathcal{X}}(A[n])) \longrightarrow 0$$

and isomorphisms:

$$\mathsf{H}(\Omega^2_{\mathcal{X}}(A[n])[-k]) = 0, \quad 1 \leqslant k \leqslant n-k-1$$

Continuing in this way, and finally applying H to the triangle $(T^{k-1}_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{k-2}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{k-2}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

and an isomorphism:

$$\mathsf{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])[-1]) = 0$$

Finally applying H to the triangle $(T^k_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^k_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{k-1}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{k-1}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

From the above exact sequences we deduce that:

$$\Omega^k\mathsf{H}(A[n]) \xrightarrow{\cong} \mathsf{H}(\Omega^k_{\mathcal{X}}(A[n])) \tag{9.16}$$

On the other hand from the cellular tower of $A[n]$ we have triangles:

$$\Omega^t_{\mathcal{X}}(A[n])[t-1] \longrightarrow \mathsf{Cell}_{t-1}(A[n]) \longrightarrow A[n] \longrightarrow \Omega^t_{\mathcal{X}}(A[n])[t], \qquad 1 \leqslant t \leqslant n \qquad (C^{t-1}_{A[n]})$$

Applying H and using that $\mathsf{H}(A[i]) = 0$, $1 \leqslant i \leqslant n-1$, and, by Proposition 4.3, $\mathsf{H}(\Omega^t_{\mathcal{X}}(A[n])[i]) = 0$, $1 \leqslant i \leqslant t$, we deduce an exact sequence

$$\mathsf{H}\big(\mathsf{Cell}_{t-1}(A[n])[-n]\big) \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^t_{\mathcal{X}}(A[n])[-n+t]\big) \longrightarrow \mathsf{H}\big(\mathsf{Cell}_{t-1}(A[n])[-n+1]\big) \longrightarrow 0$$

Since $\mathsf{Cell}_{t-1}(A[n])[-n]$ lies in $(\mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[t-1])[-n] = \mathcal{X}[-n] \star \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-n+t-1]$ which is contained in $\mathcal{X}[-n] \star \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, since $t \leqslant n$, by Lemma 9.5 we deduce short exact sequences

$$0 \longrightarrow \mathsf{H}(A) \longrightarrow \mathsf{H}\big(\Omega^t_{\mathcal{X}}(A[n])[-n+t]\big) \longrightarrow \mathsf{H}\big(\mathsf{Cell}_{t-1}(A[n])[-n+1]\big) \longrightarrow 0 \tag{9.17}$$

for $2 \leqslant t \leqslant n$. Moreover we infer isomorphisms:

$$\mathsf{H}(\Omega^t_{\mathcal{X}}(A[n])[-n+t+1]) \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_{t-1}(A[n])[-n+2])$$

$$\mathsf{H}(\Omega^t_{\mathcal{X}}(A[n])[-n+t+2]) \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_{t-1}(A[n])[-n+3])$$

$$\vdots$$

$$\mathsf{H}(\Omega^t_{\mathcal{X}}(A[n])[t-2]) \xrightarrow{\cong} \mathsf{H}(\mathsf{Cell}_{t-1}(A[n])[-1]) \tag{9.18}$$

Consider the short exact sequence (9.17) for $t = n-1$. Since $\mathsf{Cell}_{n-2}(A[n])[-n+1]$ lies in $\mathcal{X}[-n+1]\star\cdots\mathcal{X}[-1]$, it follows that $\mathsf{pd}\,\mathsf{H}(\mathsf{Cell}_{n-2}(A[n])[-n+1]) \leqslant k$. Then (9.17) gives us an isomorphism in $\underline{\mathsf{mod}}\text{-}\mathcal{X}$:

$$\Omega^k\mathsf{H}(A) \xrightarrow{\cong} \Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \tag{9.19}$$

From the triangle $(T^n_{A[n]})$, we have an exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^n_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{n-1}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

Since $\Omega^n_{\mathcal{X}}(A[n]) := X^n_{A[n]}$ lies in $\mathcal{X}$, it follows that

$$\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \xrightarrow{\cong} \Omega^2\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \tag{9.20}$$

Consider the short exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{n-2}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

induced from the triangle $(T^{n-2}_{A[n]}$. Then we have a short exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \longrightarrow \Omega\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \longrightarrow 0 \tag{9.21}$$

Setting $t = n-2$ in (9.18), we have an isomorphism $\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) \cong \mathsf{H}(\mathsf{Cell}_{n-3}(A[n])[-n+2])$. Since $\mathsf{Cell}_{n-3}(A[n])[-n+2]$ lies in $\mathcal{X}[-n+2]\star\cdots\star\mathcal{X}[-1]$, it follows that by hypothesis that $\mathsf{pd}\,\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) \leqslant k$. Hence from (9.21) we get an isomorphism $\Omega^{k+1}\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+2}\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n]))$, which since $\Omega^k\,\mathsf{mod}\text{-}\mathcal{X} = \mathsf{GProj}\,\mathsf{mod}\text{-}\mathcal{X}$, gives us an isomorphism:

$$\Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+1}\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \tag{9.22}$$

Next consider the exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \longrightarrow \mathsf{H}(X^{n-3}_{A[n]}) \longrightarrow \mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

induced from the triangle $(T^{n-3}_{A[n]})$. Then we have a short exact sequence

$$0 \longrightarrow \mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])[-1]) \longrightarrow \mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \longrightarrow \Omega\mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])) \longrightarrow 0$$

Setting $t = n-3$ in (9.18), we have an isomorphism $\mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])[-1]) \cong \mathsf{H}(\mathsf{Cell}_{n-4}(A[n])[-n+3])$. Since $\mathsf{Cell}_{n-4}(A[n])[-n+3]$ lies in $\mathcal{X}[-n+2]\star\cdots\star\mathcal{X}[-1]$, it follows that by hypothesis that $\mathsf{pd}\,\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])[-1]) \leqslant k$. Hence from the above exact sequence we obtain as above an isomorphism:

$$\Omega^k\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+1}\mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])) \tag{9.23}$$

Combining (9.22) and (9.23) we arrive at an isomorphism:

$$\Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+1}\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+2}\mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n]))$$

Continuing in this way we obtain inductively isomorphisms:

$$\begin{aligned}\Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) &\xrightarrow{\cong} \Omega^{k+1}\mathsf{H}(\Omega^{n-2}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{k+2}\mathsf{H}(\Omega^{n-3}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \cdots \\ &\xrightarrow{\cong} \Omega^{n-2}\mathsf{H}(\Omega^{k+1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{n-1}\mathsf{H}(\Omega^{k}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^{n-1}\Omega^k\mathsf{H}(A[n])\end{aligned} \tag{9.24}$$

Combining the isomorphisms (9.22)-(9.24) we have isomorphisms:

$$\begin{aligned}\Omega^k\mathsf{H}(A) &\xrightarrow{\cong} \Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])[-1]) \xrightarrow{\cong} \Omega^2\Omega^k\mathsf{H}(\Omega^{n-1}_{\mathcal{X}}(A[n])) \xrightarrow{\cong} \Omega^2\Omega^{n-1}\Omega^k\mathsf{H}(A[n]) \xrightarrow{\cong} \\ &\Omega^{n+1}\Omega^k\mathsf{H}(A[n]) \xrightarrow{\cong} \Omega^k\Omega^{n+1}\mathsf{H}(A[n])\end{aligned} \tag{9.25}$$

Since $\mathsf{H}(A)$ and $\Omega^{n+1}\mathsf{H}(A[n])$ are Gorenstein-projective, we infer an isomorphism:

$$\mathsf{H}(A) \xrightarrow{\cong} \Omega^{n+1}\mathsf{H}(A[n])$$

Hence in the stable category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ we have isomorphisms:

$$\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-k}\Omega^k\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-k}\Omega^k\Omega^{n+1}\mathsf{H}(A[n]) \xrightarrow{\cong} \Omega^{n+1}\Omega^{-k}\Omega^k\mathsf{H}(A[n])$$

We infer that:

$$\Omega^{-(n+1)}\mathsf{H}(A) \xrightarrow{\cong} \Omega^{-k}\Omega^k\mathsf{H}(A[n])$$

as required. Then by Lemma 9.6 we infer that $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau. $\square$

Note that Theorem 9.7 is a special case of Theorem 9.11, since $\mathcal{X}$ is $(n-k)$-strong and $0 \leqslant k \leqslant 1$, then as easily seen $\mathsf{H}(C) = 0$, for any $C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$.

From now and on until the end of this section we fix an $(n+1)$-Calabi-Yau triangulated category $\mathcal{T}$ over a field k, for instance $\mathcal{T} = \mathscr{C}_H^{(n+1)}$ the $(n+1)$-cluster category of a finite-dimensional hereditary k-algebra H. Let $\mathcal{X}$ be an $(n-k)$-strong $(n+1)$-cluster tilting subcategory of $\mathcal{T}$, $0 \leqslant k \leqslant n-1$, and assume that if $k \geqslant 2$, then $n \geqslant 2k-1$ and any object $\mathsf{H}(C)$, where $C \in \mathcal{X}[-n+1] \star \cdots \star \mathcal{X}[-1]$, has finite projective or injective dimension.

Corollary 9.12. *The triangulated category* $\underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X}$ *has Auslander-Reiten triangles and the Auslander-Reiten translation is given by:*

$$\tau = \Omega^{-(n+1)} \ : \ \underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X} \xrightarrow{\approx} \underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\mathcal{X}, \qquad \tau\mathsf{H}(A) = \Omega^{-(n+1)}\mathsf{H}(A) = \Omega^{-k}\Omega^{k}\mathsf{H}(\mathsf{Cell}_1(A)[n])$$

Recall that the triangulated category of singularities, in the sense of Orlov, see [24], associated to a finite-dimensional k-algebra Λ, is the Verdier quotient $\mathbf{D}^b(\mathsf{mod}\text{-}\Lambda)/\mathbf{K}^b(\mathsf{proj}\,\Lambda)$ of the bounded derived category of finite-dimensional Λ-modules by the thick subcategory of perfect complexes.

Corollary 9.13. *If* $\mathcal{X} = \mathsf{add}\,T$ *for some object* $T \in \mathcal{T}$, *then the triangulated category of singularities* $\mathbf{D}_{\mathsf{sing}}\big(\mathsf{End}_{\mathcal{T}}(T)\big)$ *associated to the k-algebra* $\mathsf{End}_{\mathcal{T}}(T)$ *is* $(n+2)$*-Calabi-Yau.*

Proof. By Theorem 9.11 the endomorphism algebra $\mathsf{End}_{\mathcal{T}}(T)$ is k-Gorenstein. Since $\mathcal{T}$ is $(n+1)$-Calabi-Yau, by Theorem 7.5, the stable triangulated category $\underline{\mathsf{Gproj}}(\mathsf{End}_{\mathcal{T}}(T))$ of finitely generated Gorenstein-projective $\mathsf{End}_{\mathcal{T}}(T)$-modules is $(n+2)$-Calabi-Yau. Then the assertion follows from the well-known fact that over a Gorenstein algebra Λ the triangulated category of singularities $\mathbf{D}_{\mathsf{sing}}(\Lambda)$ of Λ is triangle equivalent to the stable category $\underline{\mathsf{Gproj}}\,\Lambda$ of finitely generated Gorenstein-projective modules over Λ. □

Corollary 9.14. *The stable category* $\underline{\mathsf{mod}}\text{-}\underline{\mathsf{Gproj}}(\mathcal{X})$ *of coherent functors over the stable category* $\underline{\mathsf{Gproj}}(\mathcal{X})$ *of Gorenstein-projective coherent functors over* $\mathcal{X}$ *is a triangulated category which is* $(3n+5)$*-Calabi-Yau.*

Proof. By Theorem 9.11, the triangulated category $\underline{\mathsf{Gproj}}(\mathcal{X})$ is $(n+2)$-Calabi-Yau. The category of coherent functors $\mathsf{mod}\text{-}\underline{\mathsf{Gproj}}\,\mathcal{X}$ over $\underline{\mathsf{Gproj}}\,\mathcal{X}$ is Frobenius and therefore its stable category $\underline{\mathsf{mod}}\text{-}\underline{\mathsf{Gproj}}\,\mathcal{X}$ is triangulated. The the assertion follows by a result of Keller which says that if a triangulated category $\mathscr{C}$ is d-Calabi-Yau, then the stable category $\underline{\mathsf{mod}}\text{-}\mathscr{C}$ of coherent functors over $\mathscr{C}$ is $(3d-1)$-Calabi-Yau. □

Let Λ be a finite-dimensional k-algebra over a field k and assume that Λ is of finite CM-type. Let G be an additive generator of $\mathsf{Gproj}\,\Lambda$, i.e. $\mathsf{Gproj}\,\Lambda = \mathsf{add}\,G$. The endomorphism algebra $\mathsf{End}_\Lambda(G)$ of G is called the *Auslander-Cohen-Macaulay algebra* of Λ, and its stable endomorphism version $\underline{\mathsf{End}}_\Lambda(G) := \underline{\mathsf{Hom}}_\Lambda(G,G)$, is called the *stable Auslander-Cohen-Macaulay algebra* of Λ. Note that $\underline{\mathsf{End}}_\Lambda(G)$ is self-injective, see [10]. Now as a consequence of Corollary 9.14 we have the following.

Corollary 9.15. *If the cluster-tilted algebra* $\mathsf{End}_{\mathcal{T}}(T)$ *is of finite Cohen-Macaulay type and G is an additive generator of* $\mathsf{Gproj}\,\mathsf{End}_{\mathcal{T}}(T)$, *then the stable module category* $\underline{\mathsf{mod}}\text{-}\underline{\mathsf{End}}_\Lambda(G)$ *is* $(3n+5)$*-Calabi-Yau.*

Remark 9.16. In this section we considered ($(n-k)$-strong) $(n+1)$-cluster tilting subcategories $\mathcal{X}$ in triangulated categories $\mathcal{T}$ which are assumed to be d-Calabi-Yau, for $d = n+1$. By Serre duality we have $\mathsf{D}\mathcal{T}(\mathcal{X},\mathcal{X}) \cong \mathcal{T}(\mathcal{X},\mathcal{X}[d])$. If $0 < d < n+1$, it follows that $\mathsf{D}\mathcal{T}(\mathcal{X},\mathcal{X}) = 0$ and therefore $\mathcal{X} = 0$. If $d = n+k$, where $2 \leqslant k \leqslant n-1$, and $\mathcal{X}$ is $(n-k)$-strong with $n \geqslant 2k-1$, then we know by Corollary 7.4 that $\mathcal{X}[n+k] \subseteq \mathcal{X}[k-1] \star \mathcal{X}[k] \star \cdots \star \mathcal{X}[2k-1]$. Since $2k-1 \leqslant n$, this easily implies that $\mathcal{T}(\mathcal{X},\mathcal{X}[n+k]) = 0$ and then again $\mathcal{X} = 0$. Therefore a d-Calabi-Yau triangulated category $\mathcal{T}$ may contain a non-trivial $(n+1)$-cluster tilting subcategory, only if $d \geqslant n+1$, and may contain a non-trivial $(n-k)$-strong $(n+1)$-cluster tilting subcategories, only if $d \notin \{1,2,\cdots,n,n+2,\cdots,2n\}$.

10. Global Dimension of Non-Stable Cluster Tilting Subcategories

In this section we are interested in the non-stable versions of the results of the previous sections and in particular we are concerned with the estimation of the global dimension of the category of coherent functors over a cluster-tilting subcategory of an abelian category induced by a cluster-tilting subcategory of its Gorenstein-projectives.

Let $\mathscr{A}$ be an abelian category and $\mathcal{M} \subseteq \mathscr{A}$ a full subcategory. In analogy with the triangulated case define

$$\mathcal{M}_n^{\perp} = \big\{A \in \mathscr{A} \ \mid \ \mathsf{Ext}^k(\mathcal{M},A) = 0, \ 1 \leqslant k \leqslant n\big\} \qquad \text{and} \qquad {}^{\perp}_{n}\mathcal{M} = \big\{A \in \mathscr{A} \ \mid \ \mathsf{Ext}^k(A,\mathcal{M}) = 0, \ 1 \leqslant k \leqslant n\big\}$$

Then $\mathcal{M}$ is called *n-rigid* if $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$ or equivalently $\mathcal{M} \subseteq {}^{\perp}_{n}\mathcal{M}$.

Lemma 10.1. *Let $\mathscr{A}$ be an abelian category with enough projectives. Then for any object $A \in \mathscr{A}$ with finite projective dimension we have $A = 0$ if and only if $\mathsf{Ext}^k(A, P) = 0$, $\forall P \in \mathsf{Proj}\,\mathscr{A}$, $0 \leqslant k \leqslant \mathsf{pd}\,A$.*

Proof. Let $\mathsf{pd}\,A = n$ and let $0 \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow A \longrightarrow 0$ be a projective resolution of A. Then $0 = \mathsf{Ext}^n(A, P^n) \cong \mathsf{Ext}^1(\Omega^{n-1}A, P^n)$, so the sequence $0 \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \Omega^{n-1}A \longrightarrow 0$ splits and therefore $\Omega^{n-1}A$ is projective. Then $\mathsf{Ext}^1(\Omega^{n-2}A, \Omega^{n-1}A) \cong \mathsf{Ext}^{n-1}(A, \Omega^{n-1}A) = 0$, so $0 \longrightarrow \Omega^{n-1}A \longrightarrow P^{n-2} \longrightarrow \Omega^{n-2}A \longrightarrow 0$ splits and therefore $\Omega^{n-2}A$ is projective. Continuing in this way we see that $\Omega^1 A$ is projective. Then $0 = \mathsf{Ext}^1(A, \Omega^1 A) = 0$, so the sequence $0 \longrightarrow \Omega^1 A \longrightarrow P^0 \longrightarrow A \longrightarrow 0$ splits and therefore A is projective. Then $A = 0$ since $\mathscr{A}(A, A) = 0$. $\square$

From now on, let $\mathscr{A}$ be an abelian category with enough projectives and assume that $\mathcal{M}$ is a contravariantly finite subcategory of $\mathscr{A}$ such that $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{M}$.

Since $\mathcal{M}$ is contravariantly finite in $\mathscr{A}$, the category $\mathsf{mod}\text{-}\mathcal{M}$ is abelian. We consider the, fully faithful since $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{M}$, functor

$$\mathbb{H} \ : \ \mathscr{A} \ \longrightarrow \ \mathsf{mod}\text{-}\mathcal{M}, \quad \mathbb{H}(A) = \mathscr{A}(-, A)|_{\mathcal{M}}$$

Then $\mathbb{H}$ admits an exact left adjoint which is given by the restriction functor $\mathbb{R} : \mathsf{mod}\text{-}\mathcal{M} \longrightarrow \mathsf{mod}\text{-}\mathsf{Proj}\,\mathscr{A} \approx \mathscr{A}$, induced by the inclusion $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{M}$. Clearly $\mathsf{Ker}\,\mathbb{R}$ consists of all coherent functors $F : \mathcal{M}^{\mathsf{op}} \longrightarrow \mathscr{A}b$ which admit a projective presentation $\mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0$ where the map $M^1 \longrightarrow M^0$ is an epimorphism in $\mathscr{A}$. Further $\mathsf{Ker}\,\mathbb{R}$ coincides with the full subcategory of $\mathsf{mod}\text{-}\mathcal{M}$ consisting of all coherent functors over $\mathcal{M}$ vanishing on the projectives and is equivalent to the category $\mathsf{mod}\text{-}\underline{\mathcal{M}}$ of coherent functors over the stable category $\underline{\mathcal{M}}^{\mathsf{op}} \longrightarrow \mathscr{A}b$ and is Hence we have a short exact sequence of abelian categories

$$0 \ \longrightarrow \ \mathsf{mod}\text{-}\underline{\mathcal{M}} \ \longrightarrow \ \mathsf{mod}\text{-}\mathcal{M} \ \longrightarrow \mathscr{A} \ \longrightarrow 0 \tag{10.1}$$

Let $F \in \mathsf{mod}\text{-}\mathcal{M}$, and fix throughout a projective presentation $\mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0$ of F. Then we have an exact sequence

$$0 \ \longrightarrow \ \mathbb{H}(A_F) \ \longrightarrow \ \mathbb{H}(M^1) \ \longrightarrow \ \mathbb{H}(M^0) \ \longrightarrow \ F \ \longrightarrow \ 0 \tag{10.2}$$

where $A_F = \mathsf{Ker}(M^1 \longrightarrow M^0)$ and then $\Omega(F) = \mathsf{Im}(\mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0))$ and $\mathbb{H}(A_F) = \Omega^2(F)$.

Let $A \in \mathscr{A}$. Then there exists a long exact sequence, called an $\mathcal{M}$-*resolution of* A,

$$\cdots \longrightarrow M_A^n \longrightarrow M_A^{n-1} \longrightarrow \cdots \longrightarrow M_A^1 \longrightarrow M_A^0 \longrightarrow A \longrightarrow 0 \tag{10.3}$$

such that its image under $\mathbb{H}$ is a projective resolution of $\mathbb{H}(A)$. This is defined as the Yoneda composition of short exact sequences $0 \longrightarrow K_A^{i+1} \longrightarrow M_A^i \longrightarrow K_A^i \longrightarrow 0$ constructed inductively, where each map $M_A^i \longrightarrow K_A^i$ is a right $\mathcal{M}$-approximation of K^i, $\forall i \geqslant 0$, and $K_A^0 = A$. Note that the right $\mathcal{M}$-approximations are epics since $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{M}$. It follows that projective resolutions of $F \in \mathsf{mod}\text{-}\mathcal{M}$ are of the form

$$\cdots \longrightarrow \mathbb{H}(M_{A_F}^n) \longrightarrow \mathbb{H}(M_{A_F}^{n-1}) \longrightarrow \cdots \longrightarrow \mathbb{H}(M_{A_F}^0) \longrightarrow \mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0 \tag{10.4}$$

where $\mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0$ and $A_F = \mathsf{Ker}(M^1 \longrightarrow M^0)$.

Lemma 10.2. (i) *For any object $F \in \mathsf{mod}\text{-}\mathcal{M}$ the following are equivalent:*
- (a) $F \in \mathsf{mod}\text{-}\underline{\mathcal{M}}$
- (b) $\mathsf{Ext}^k(F, \mathbb{H}(A)) = 0$, $0 \leqslant k \leqslant 1$, $\forall A \in \mathscr{A}$.

If $F \in \mathsf{mod}\text{-}\underline{\mathcal{M}}$ and $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$, then there are isomorphisms:

$$F \ \cong \ \mathsf{Ext}^1(-, A_F)|_{\mathcal{M}} \ \cong \ \mathsf{Ext}^2(-, K_{A_F}^1)|_{\mathcal{M}} \ \cong \ \cdots \ \cong \ \mathsf{Ext}^{n-1}(-, K_{A_F}^{n-2})|_{\mathcal{M}} \ \cong \ \mathsf{Ext}^n(-, K_{A_F}^{n-1})|_{\mathcal{M}}$$

(ii) *For any object $B \in \mathscr{A}$, and any $n \geqslant 1$, the following are equivalent:*
- (a) $B \in \mathcal{M}_n^{\top}$.
- (b) *For any $A \in \mathscr{A}$, there are isomorphisms:*

$$\mathsf{Ext}^k(A, B) \ \xrightarrow{\cong} \ \mathsf{Ext}^k(\mathbb{H}(A), \mathbb{H}(B)), \qquad 1 \leqslant k \leqslant n$$

(iii) *The following are equivalent:*
- (a) $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$.
- (b) *For any $F \in \mathsf{mod}\text{-}\underline{\mathcal{M}} \subseteq \mathsf{mod}\text{-}\mathcal{M}$ and any $X \in \mathcal{M}$, we have:*

$$\mathsf{Ext}^k(F, \mathbb{H}(X)) = 0, \quad 0 \leqslant k \leqslant n+1$$

Proof. (i) (b) $\Rightarrow$ (a) Since F lies in $\mathsf{mod}\text{-}\underline{\mathcal{M}}$, the map $M^1 \longrightarrow M^0$ in (10.2) is epic and we have an exact sequence $0 \longrightarrow A_F \longrightarrow M^1 \longrightarrow M^0 \longrightarrow 0$ which, for any $B \in \mathscr{A}$, induces a long exact sequence $0 \longrightarrow \mathscr{A}(M^0, B) \longrightarrow \mathscr{A}(M^1, B) \longrightarrow \mathscr{A}(A, B) \longrightarrow \mathsf{Ext}^1(M^0, B) \longrightarrow \cdots$. Since $\mathbb{H}$ is fully faithful, the monomorphism $\mathscr{A}(M^0, B) \longrightarrow \mathscr{A}(M^1, B)$ is isomorphic to the map $(\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(M^1), \mathbb{H}(B))$ whose kernel is isomorphic to $(F, \mathbb{H}(B))$. We infer that $(F, \mathbb{H}(B)) = 0$. This implies that we have an exact sequence $0 \longrightarrow (\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\Omega(F), \mathbb{H}(B)) \longrightarrow \mathsf{Ext}^1(F, \mathbb{H}(B)) \longrightarrow 0$. On the other hand we have a long exact sequence $0 \longrightarrow (\Omega(F), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(M^1), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(A), \mathbb{H}(B)) \longrightarrow \mathsf{Ext}^1(F, \mathbb{H}(B)) \longrightarrow 0$. Since $\mathbb{H}$ is fully faithful, the map $\mathscr{A}(M^1, B) \longrightarrow \mathscr{A}(A, B)$ is isomorphic to the map $(\mathbb{H}(M^1), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(A), \mathbb{H}(B))$. By diagram chasing this implies that the map $(\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\Omega(F), \mathbb{H}(B))$ is an isomorphism. Hence $\mathsf{Ext}^1(F, \mathbb{H}(B)) = 0$ as required.

(a) $\Rightarrow$ (b) It suffices to show that the map $M^1 \longrightarrow M^0$ in (10.2) is epic. Let $C = \mathsf{Coker}(M^1 \longrightarrow M^0)$. Then $\mathbb{H}(M^0) \longrightarrow \mathbb{H}(C)$ factorizes through F. Since by hypothesis $(F, \mathbb{H}(B)) = 0$, $\forall B \in \mathscr{A}$, it follows that the map $\mathbb{H}(M^0) \longrightarrow \mathbb{H}(C)$, or equivalently the map $M^0 \longrightarrow C$ is zero. Hence $C = 0$.

Assume now that $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$. Then the map $M^1 \longrightarrow M^0$ inducing the exact sequence (10.2) is an epimorphism, so we have a short exact sequence $0 \longrightarrow A_F \longrightarrow M^1 \longrightarrow M^0 \longrightarrow 0$ in $\mathscr{A}$. Since $\mathsf{Ext}^1(-, M^1)|_{\mathcal{M}} = 0$, it follows that $F \cong \mathsf{Ext}^1(-, A_F)|_{\mathcal{M}}$. Now consider an $\mathcal{M}$-resolution $\cdots \longrightarrow M^1_{A_F} \longrightarrow M^0_{A_F} \longrightarrow A_F \longrightarrow 0$ which is build from short exact sequences $0 \longrightarrow K^i_{A_F} \longrightarrow M^{i-1}_{A_F} \longrightarrow K^{i-1}_{A_F} \longrightarrow 0$, where the last map is a right $\mathcal{M}$-approximation. Using that $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$, as above we have isomorphisms $\mathsf{Ext}^1(-, A_F)|_{\mathcal{M}} \cong \mathsf{Ext}^2(-, K^1_{A_F})|_{\mathcal{M}} \cong \cdots \cong \mathsf{Ext}^n(-, K^{n-2}_{A_F})|_{\mathcal{M}} \cong \mathsf{Ext}^n(-, K^{n-1}_{A_F})|_{\mathcal{M}}$ and the assertion follows.

(ii) (b) $\Rightarrow$ (a) Setting $A = X \in \mathcal{M}$, and using that $\mathbb{H}(X)$ is projective in $\mathsf{mod}\text{-}\mathcal{M}$, we infer that $\mathsf{Ext}^k(X, B) = 0$, $1 \leqslant k \leqslant n$. Hence B lies in $\mathcal{M}_n^{\perp}$.

(a) $\Rightarrow$ (b) First let $n = 1$. Consider the short exact sequence $0 \longrightarrow K^1 \longrightarrow M^0 \longrightarrow A \longrightarrow 0$. Then we have an exact sequence $0 \longrightarrow \mathbb{H}(K^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow \mathbb{H}(A) \longrightarrow 0$. Using that $\mathsf{Ext}^1(M^0, B) = 0$, we then have an exact sequence $0 \longrightarrow \mathscr{A}(A, B) \longrightarrow \mathscr{A}(M^0, B) \longrightarrow \mathscr{A}(K^1, B) \longrightarrow \mathsf{Ext}^1(A, B) \longrightarrow 0$. Since $\mathbb{H}(M^0)$ is projective, we also have an exact sequence $0 \longrightarrow (\mathbb{H}(A), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(K^1), \mathbb{H}(B)) \longrightarrow \mathsf{Ext}^1(\mathbb{H}(A), \mathbb{H}(B)) \longrightarrow 0$. Since the left exact functor $\mathbb{H}$ is fully faithful, the sequences $0 \longrightarrow \mathscr{A}(A, B) \longrightarrow \mathscr{A}(M^0, B) \longrightarrow \mathscr{A}(K^1, B)$ and $0 \longrightarrow (\mathbb{H}(A), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(M^0), \mathbb{H}(B)) \longrightarrow (\mathbb{H}(K^1), \mathbb{H}(B))$ are isomorphic. It follows that there is an isomorphism $\mathsf{Ext}^1(A, B) \cong \mathsf{Ext}^1(\mathbb{H}(A), \mathbb{H}(B))$. Applying $\mathscr{A}(-, B)$ to the short exact sequences $0 \longrightarrow K^i \longrightarrow M^{i-1} \longrightarrow K^{i-1} \longrightarrow 0$, $1 \leqslant i \leqslant n-1$, and using $\mathsf{Ext}^k(\mathcal{M}, \mathcal{M}) = 0$, $1 \leqslant k \leqslant n$, it is easy to see by dimension shift that we have isomorphisms: $\mathsf{Ext}^n(A, B) \cong \mathsf{Ext}^{n-1}(K^1, B) \cong \cdots \cong \mathsf{Ext}^1(K^{n-1}, B)$. Then considering the short exact sequence $0 \longrightarrow K^n \longrightarrow M^{n-1} \longrightarrow K^{n-1} \longrightarrow 0$, we see, as in the case $n = 1$, that we have an isomorphism $\mathsf{Ext}^n(A, B) \cong \mathsf{Ext}^1(K^{n-1}, B) \cong \mathsf{Ext}^1(\mathbb{H}(K^{n-1}), \mathbb{H}(B))$. Since clearly $\mathbb{H}(K^{n-1}) = \Omega^{n-1}\mathbb{H}(A)$, we infer that we have an isomorphism $\mathsf{Ext}^n(A, B) \cong \mathsf{Ext}^1(\mathbb{H}(A), \mathbb{H}(B))$ and the assertion follows by induction.

(iii) (b) $\Rightarrow$ (a) Let $X \in \mathcal{M}$ and consider the functor $(-, \underline{X}) : \mathcal{M}^{\mathsf{op}} \longrightarrow \mathscr{Ab}$, $M' \longmapsto \underline{\mathcal{M}}(M', X)$. If $(\dagger) : 0 \longrightarrow \Omega(X) \longrightarrow P_X \longrightarrow X \longrightarrow 0$ is an exact sequence, where P_X is projective, then it is easy to see that we have an exact sequence $\mathbb{H}(P_X) \longrightarrow \mathbb{H}(X) \longrightarrow (-, \underline{X}) \longrightarrow 0$, so $(-, \underline{X})$ lies in $\mathsf{mod}\text{-}\underline{\mathcal{M}} \subseteq \mathsf{mod}\text{-}\mathcal{M}$, and the exact sequence (10.2) for $F = (-, \underline{X})$ takes the form

$$0 \longrightarrow \mathbb{H}(\Omega(X)) \longrightarrow \mathbb{H}(P_X) \longrightarrow \mathbb{H}(X) \longrightarrow (-, \underline{X}) \longrightarrow 0$$

Let $G = \mathsf{Im}(\mathbb{H}(P_X) \longrightarrow \mathbb{H}(X))$. Clearly then $\mathsf{Ext}^2((-, \underline{X}), \mathbb{H}(X)) \cong \mathsf{Ext}^1(G, \mathbb{H}(\mathcal{M}) \cong \mathsf{Coker}[\mathscr{A}(P_X, \mathcal{M}) \longrightarrow \mathscr{A}(\Omega(X), \mathcal{M})] \cong \mathsf{Ext}^1(X, \mathcal{M})$. It follows by hypothesis that $\mathsf{Ext}^1(X, \mathcal{M}) = 0$. Next for $2 \leqslant k \leqslant n+1$ we have $0 = \mathsf{Ext}^{k+1}((-, \underline{X}), \mathbb{H}(\mathcal{M})) \cong \mathsf{Ext}^{k-1}(\mathbb{H}(\Omega(X)), \mathbb{H}(\mathcal{M}))$. Since $\mathsf{Ext}^1(X, \mathcal{M}) = 0$ and X is arbitrary, it follows from (ii) that for any object $A \in \mathcal{M}$ we have an isomorphism $\mathsf{Ext}^1(A, \mathcal{M}) \cong \mathsf{Ext}^1(\mathbb{H}(A), \mathbb{H}(\mathcal{M}))$, in particular $\mathsf{Ext}^1(\Omega(X), \mathcal{M}) \cong \mathsf{Ext}^1(\mathbb{H}(\Omega(X)), \mathbb{H}(\mathcal{M}))$. Since the last space is zero, we have $\mathsf{Ext}^1(\Omega(X), \mathcal{M}) \cong \mathsf{Ext}^2(X, \mathcal{M}) = 0$. It follows that $\mathsf{Ext}^2(\mathcal{M}, \mathcal{M}) = 0$ and therefore $\mathcal{M} \subseteq \mathcal{M}_2^{\perp}$. Then by (ii) we have an isomorphism $\mathsf{Ext}^3(X, \mathcal{M}) \cong \mathsf{Ext}^2(\Omega(X), \mathcal{M}) \cong \mathsf{Ext}^2(\mathbb{H}(\Omega(X)), \mathbb{H}(\mathcal{M}))$. Since the last space is zero we infer that $\mathsf{Ext}^3(X, \mathcal{M}) = 0$ and therefore $\mathcal{M} \subseteq \mathcal{M}_3^{\perp}$. Continuing in this way we have finally that $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$ as required.

(a) $\Rightarrow$ (b) From (10.2) we see that $\mathsf{Ext}^{k+2}(F, \mathbb{H}(X) \cong \mathsf{Ext}^k(\mathbb{H}(A_F), \mathbb{H}(X))$, $\forall k \geqslant 1$. Since $\mathcal{M} \subseteq \mathcal{M}_n^{\perp}$, by (ii) we infer isomorphisms $\mathsf{Ext}^{k+2}(F, \mathbb{H}(X)) \cong \mathsf{Ext}^k(A_F, X)$, $1 \leqslant k \leqslant n$. Now since $F \in \mathsf{mod}\text{-}\underline{\mathcal{M}}$, we have a short exact sequence $0 \longrightarrow A_F \longrightarrow M^1 \longrightarrow M^0 \longrightarrow 0$. Applying $\mathscr{A}(-, X)$, with $X \in \mathcal{M}$, the induced long exact sequence gives $\mathsf{Ext}^k(A_F, X) = 0$, $1 \leqslant k \leqslant n-1$. We infer that $\mathsf{Ext}^{k+2}(F, \mathbb{H}(X)) \cong \mathsf{Ext}^k(A_F, X)$, $1 \leqslant k \leqslant n-1$. Hence using (i) we have $\mathsf{Ext}^k(F, \mathbb{H}(X)) = 0$, $0 \leqslant k \leqslant 1$ and $3 \leqslant k \leqslant n+1$. Finally applying $\mathbb{H}(X)$ to the short exact sequence $0 \longrightarrow \mathbb{H}(A_F) \longrightarrow \mathbb{H}(M^1) \longrightarrow \Omega(F) \longrightarrow 0$ we have an exact sequence $(\mathbb{H}(M^1), \mathbb{H}(X)) \longrightarrow (\mathbb{H}(A), \mathbb{H}(X)) \longrightarrow \mathsf{Ext}^2(F, \mathbb{H}(X) \longrightarrow 0$. Since the map $(\mathbb{H}(M^1), \mathbb{H}(X)) \longrightarrow$

$(\mathbb{H}(A),\mathbb{H}(X))$ is isomorphic to the map $(M^1,X)\longrightarrow(A,X)$ which is an epimorphism since $\mathsf{Ext}^1(M^0,X)=0$, we infer that $\mathsf{Ext}^2(F,\mathbb{H}(X))=0$. Hence $\mathsf{Ext}^k(F,\mathbb{H}(X))=0$, $0\leqslant k\leqslant n+1$, as required. □

Proposition 10.3. *If* $\mathsf{Proj}\,\mathscr{A}\subsetneqq\mathcal{M}\subseteq\mathcal{M}_n^{\perp}$, *then* $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}\geqslant n+2$. *Moreover if* $\mathcal{M}_n^{\perp}=\mathcal{M}$, *then* ${}_n^{\perp}\mathcal{M}=\mathcal{M}$ *and* $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$.

Proof. Since $\mathcal{M}\neq\mathsf{Proj}\,\mathscr{A}$, we have $\mathsf{mod}\text{-}\underline{\mathcal{M}}\neq 0$ so there exists $0\neq F\in\mathsf{mod}\text{-}\underline{\mathcal{M}}$. If $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}\leqslant n+1$, then by Lemma 1.2(iii) we have $\mathsf{Ext}^k(F,\mathbb{H}(X))=0$, $0\leqslant k\leqslant n+1$, and then by Lemma 10.1 it follows that $F=0$. This contradiction shows that $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}\geqslant n+2$. Now assume that $\mathcal{M}_n^{\perp}=\mathcal{M}$. To show that $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$, it suffices to show that $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}\leqslant n+2$. By using the exact sequence (10.2), this holds if $\mathsf{pd}\,\mathbb{H}(A)\leqslant n$, $\forall A\in\mathscr{A}$. Since the exact sequence (10.3) becomes a projective resolution of $\mathbb{H}(A)$ after applying $\mathbb{H}$, it suffices to show that $\mathbb{H}(K^n)$ is projective, or equivalently that $K^n\in\mathcal{M}$. Since $\mathcal{M}_n^{\perp}=\mathcal{M}$, it suffices to show that $K^n\in\mathcal{M}_n^{\perp}$. Consider the extensions $0\longrightarrow K^i\longrightarrow M^{i-1}\longrightarrow K^{i-1}\longrightarrow 0$, where each map $M^{i-1}\longrightarrow K^{i-1}$ is a right $\mathcal{M}$-approximation of K^{i-1} for $0\leqslant i\leqslant n$ and $K^0:=A$. Since $\mathsf{Ext}^k(\mathcal{M},\mathcal{M})=0$, $1\leqslant k\leqslant n$, we have clearly $\mathsf{Ext}^1(\mathcal{M},K^i)=0$, $1\leqslant i\leqslant n$. In particular $\mathsf{Ext}^1(\mathcal{M},K^n)=0$ and we have an isomorphism $\mathsf{Ext}^2(\mathcal{M},K^n)\cong\mathsf{Ext}^1(\mathcal{M},K^{n-1})=0$, hence $K^n\in\mathcal{M}_2^{\perp}$. Continuing in this way we have finally isomorphisms $\mathsf{Ext}^n(\mathcal{M},K^n)\cong\mathsf{Ext}^{n-1}(\mathcal{M},K^{n-1})\cong\cdots\cong\mathsf{Ext}^1(\mathcal{M},K^1)=0$. Hence $K^n\in\mathcal{M}_n^{\perp}=\mathcal{M}$ and therefore $\mathsf{pd}\,\mathbb{H}(A)\leqslant n$. We conclude that $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$.

Now let $A\in{}_n^{\perp}\mathcal{M}$. By applying $\mathscr{A}(-,\mathcal{M})$ to the extension $0\longrightarrow K^1\longrightarrow M^0\longrightarrow A\longrightarrow 0$, we have $\mathsf{Ext}^i(K^1,\mathcal{M})=0$, $1\leqslant i\leqslant n-1$. Using this and applying $\mathscr{A}(-,\mathcal{M})$ to the extension $0\longrightarrow K^2\longrightarrow M^1\longrightarrow K^1\longrightarrow 0$, we have $\mathsf{Ext}^i(K^2,\mathcal{M})=0$, $1\leqslant i\leqslant n-2$. Continuing in this way we finally have $\mathsf{Ext}^1(K^{n-1},\mathcal{M})=0$. Since $\mathsf{pd}\,\mathbb{H}(A)\leqslant n$, we have $K^n\in\mathcal{M}$ and therefore the extension $0\longrightarrow K^n\longrightarrow M^1\longrightarrow K^{n-1}\longrightarrow 0$ splits, hence $K^{n-1}\in\mathcal{M}$. Since $\mathsf{Ext}^1(K^{n-2},\mathcal{M})=0$, the extension $0\longrightarrow K^{n-1}\longrightarrow M^1\longrightarrow K^{n-2}\longrightarrow 0$ splits and therefore $K^{n-2}\in\mathcal{M}$. Continuing in this way we see the objects K^i lie in $\mathcal{M}$ and therefore since $\mathsf{Ext}^1(A,\mathcal{M})=0$, we infer that the extension that the extension $0\longrightarrow K^1\longrightarrow M^0\longrightarrow A\longrightarrow 0$ splits. Then $A\in\mathcal{M}$ as a direct summand of M^0. We conclude that $A\in\mathcal{M}$, i.e. ${}_n^{\perp}\mathcal{M}=\mathcal{M}$. □

Corollary 10.4. *Let* $\mathscr{A}$ *be an abelian category with enough projectives and enough injectives. Let* $\mathcal{M}$ *be a functorially finite subcategory of* $\mathscr{A}$. *Then the following are equivalent.*

(i) $\mathsf{Proj}\,\mathscr{A}\subsetneqq\mathcal{M}$ *and* $\mathcal{M}_n^{\perp}=\mathcal{M}$.
(ii) $\mathsf{Inj}\,\mathscr{A}\subsetneqq\mathcal{M}$ *and* ${}_n^{\perp}\mathcal{M}=\mathcal{M}$.
(iii) $\mathcal{M}$ *is n-rigid* , $\mathsf{Proj}\,\mathscr{A}\subsetneqq\mathcal{M}$ *or* $\mathsf{Inj}\,\mathscr{A}\subsetneqq\mathcal{M}$ *and* $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$.

If (i) *holds, then:* $\mathcal{M}\subseteq\mathsf{GProj}\,\mathscr{A}$ *if and only if* $\mathscr{A}$ *is Frobenius if and only if* $\mathcal{M}\subseteq\mathsf{GInj}\,\mathscr{A}$. *If this is the case,* $\mathscr{A}$ *is Krull-Schmidt and* $\mathcal{M}$ *is of finite representation type, then:*

$$\mathsf{rep.dim}\,\mathscr{A}\ \leqslant\ n+2$$

Proof. (i) ⇒ (ii), (iii) By Proposition 10.3 we have $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$, ${}_n^{\perp}\mathcal{M}=\mathcal{M}$ and $\mathsf{Inj}\,\mathscr{A}\subseteq\mathcal{M}$. If $\mathsf{Inj}\,\mathscr{A}=\mathcal{M}$, then clearly $\mathscr{A}=\mathcal{M}$, hence $\mathscr{A}=\mathsf{Proj}\,\mathscr{A}=\mathcal{M}$ since then $\mathscr{A}$ is semisimple. Hence $\mathsf{Inj}\,\mathscr{A}\neq\mathcal{M}$. The implication (ii) ⇒ (i) follows by duality and is left to the reader.

(iii) ⇒ (ii) Let $A\in{}_n^{\perp}\mathcal{M}$. Since $\mathsf{gl.dim\,mod}\text{-}\mathcal{M}=n+2$, it follows that $\mathsf{pd}\,\mathbb{H}(A)\leqslant n$ and therefore we have an $\mathcal{M}$-resolution $0\longrightarrow M_A^n\longrightarrow M_A^{n-1}\longrightarrow\cdots\longrightarrow M_A^1\longrightarrow M_A^0\longrightarrow A\longrightarrow 0$ of A of length $\leqslant n$. Applying to the extension $0\longrightarrow K_A^1\longrightarrow M^0\longrightarrow A\longrightarrow 0$ the functor $\mathscr{A}(-,\mathcal{M})$, we see that $K_A^1\in{}_{n-1}^{\perp}\mathcal{M}$ and then by induction $K_A^{n-i}\in{}_i^{\perp}\mathcal{M}$, $1\leqslant i\leqslant n-1$. In particular since $K_A^{n-1}\in{}^{\perp}{}_1\mathcal{M}$, the extension $0\longrightarrow M_A^n\longrightarrow M_A^{n-1}\longrightarrow K_A^{n-1}\longrightarrow 0$ splits and therefore $K_A^{n-1}\in\mathcal{M}$. Since $K_A^{n-2}\in{}_2^{\perp}\mathcal{M}$, it follows that the extension $0\longrightarrow K_A^{n-1}\longrightarrow M_A^{n-2}\longrightarrow K_A^{n-2}\longrightarrow 0$ splits and therefore $K_A^{n-2}\in\mathcal{M}$. Continuing in this way we see that the extension $0\longrightarrow K_A^1\longrightarrow M_A^0\longrightarrow A\longrightarrow 0$ splits and therefore $A\in\mathcal{M}$. We infer that ${}_n^{\perp}\mathcal{M}=\mathcal{M}$.

Assume now that one of the equivalent conditions (i)-(iii) hold. If $\mathscr{A}$ is Frobenius, then $\mathscr{A}=\mathsf{GProj}\,\mathscr{A}$ and then $\mathcal{M}\subseteq\mathsf{GProj}\,\mathscr{A}$. Conversely assume that $\mathcal{M}\subseteq\mathsf{GProj}\,\mathscr{A}$. Since any object of $\mathscr{A}$ admits an $\mathcal{M}$-resolution of lenght $\leqslant n$, it follows that $\mathscr{A}$ is Gorenstein and $\mathsf{G\text{-}dim}\,\mathscr{A}\leqslant n$. Since $\mathcal{M}$ contains the injectives, it follows that any injective object is Gorenstein-projective. Since $\mathsf{spli}\,\mathscr{A}<\infty$ it follows that any injective object has finite projective dimension and therefore any injective object is projective since it is Gorenstein-projective. Now if P is a projective object, then let $(\dagger):0\longrightarrow P\longrightarrow I\longrightarrow A\longrightarrow 0$ be exact, where I is injective. Applying $\mathscr{A}(\mathcal{M},-)$, we have directly that $\mathscr{A}\in\mathcal{M}_{n-1}^{\perp}$ and there is an isomorphism $\mathsf{Ext}^n(\mathcal{M},A)\cong\mathsf{Ext}^{n+1}(\mathcal{M},P)$. However since $\mathsf{silp}\,\mathscr{A}=\mathsf{G\text{-}dim}\,\mathscr{A}\leqslant n$, we infer that $\mathsf{id}\,P\leqslant n$, so $\mathsf{Ext}^{n+1}(\mathcal{M},P)=0$. Hence $\mathsf{Ext}^n(\mathcal{M},A)=0$ and therefore $A\in\mathcal{M}_n^{\perp}=\mathcal{M}$. Clearly then the extension $(\dagger)$ splits, so P is injective as a direct summand of I. Hence $\mathscr{A}$ is Frobenius. If moreover $\mathscr{A}$ is Krull-Schmidt and $\mathcal{M}=\mathsf{add}\,M$, then $\mathsf{add}\,M$ is functorially finite in $\mathscr{A}$ and M contains as a direct summand a projective generator and an injective cogenerator of $\mathscr{A}$. Since $\mathsf{mod}\text{-}\mathsf{End}_{\mathscr{A}}(M) \approx \mathsf{mod}\text{-}\mathcal{M}$, it follows that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathsf{End}_{\mathscr{A}}(M) = n+2$ and therefore $\mathsf{rep.dim}\,\mathscr{A} \leqslant n+2$. $\square$

From now on we assume that the abelian category $\mathscr{A}$ has enough projectives. We consider the stable category $\underline{\mathsf{GProj}}\,\mathscr{A}$ of Gorenstein-projective objects of $\mathscr{A}$ modulo projectives as a triangulated category with suspension functor Ω^{-1}. For any full subcategory $\underline{\mathcal{X}}$ of $\underline{\mathsf{GProj}}\,\mathscr{A}$, we denote by $\mathcal{M} = \pi^{-1}\underline{\mathcal{X}}$ the pre-image of $\underline{\mathcal{X}}$ under the projection functor $\pi : \mathsf{GProj}\,\mathscr{A} \longrightarrow \underline{\mathsf{GProj}}\,\mathscr{A}$. Note that $\mathsf{Proj}\,\mathscr{A} \subseteq \mathcal{M} \subseteq \mathsf{GProj}\,\mathscr{A}$.

Lemma 10.5. (i) *$\underline{\mathcal{X}}$ is contravariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ if and only if $\mathcal{M}$ is contravariantly finite in $\mathsf{GProj}\,\mathscr{A}$.*
(ii) *If $\mathsf{Proj}\,\mathscr{A}$ is covariantly finite in $\mathscr{A}$, then $\underline{\mathcal{X}}$ is covariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ if and only if $\mathcal{M}$ is covariantly finite in $\mathsf{GProj}\,\mathscr{A}$.*
(iii) *$\underline{\mathcal{X}}_n^{\top} = \underline{\mathcal{X}}$ if and only if $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$.*
(iv) *If $\mathscr{A}$ is Gorenstein, then:*
  (a) *$\underline{\mathcal{X}}$ is contravariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ if and only if $\mathcal{M}$ is contravariantly finite in $\mathscr{A}$.*
  (b) *If If $\mathsf{Proj}\,\mathscr{A}$ is covariantly finite in $\mathscr{A}$, then $\underline{\mathcal{X}}$ is covariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ if and only if $\mathcal{M}$ is covariantly finite in $\mathscr{A}$.*

Proof. (i) First let $\underline{\mathcal{X}}$ be contravariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ and let $G \in \mathsf{GProj}\,\mathscr{A}$. Let $\underline{f_G} : \underline{X_G} \longrightarrow \underline{G}$ be a right $\underline{\mathcal{X}}$-approximation of $\underline{G}$. Let M_G in $\mathcal{M}$ be such that $\underline{M_G} = \underline{X_G}$. Then we have a map $f_G : M_G \longrightarrow G$ which may chosen to be an epimorphism since $\mathcal{M}$ contains the projectives. If $\alpha : M' \longrightarrow G$ is a map, where $M' \in \mathcal{M}$, then we have a factorization $\underline{\alpha} = \underline{\rho} \circ \underline{f_G}$ for some map $\rho : \underline{M'} \longrightarrow \underline{X_G}$. Hence we have factorization $\alpha - \rho \circ f = \kappa \circ \varepsilon$, where $\kappa : M' \longrightarrow P$ and $\varepsilon : P \longrightarrow G$ and P is projective. Since f_G is an epimorphism, there exists a map $\lambda : P \longrightarrow M_G$ such that $\lambda \circ f_G = \varepsilon$. Then $\alpha = (\rho + \kappa \circ \lambda) \circ f_G$, i.e. the map $\underline{f_G}$ is a right $\mathcal{M}$-approximation. Conversely if $\mathcal{M}$ is contravariantly finite and $f : M_G \longrightarrow G$ is a right $\mathcal{M}$-approximation of $G \in \mathsf{GProj}\,\mathscr{A}$, then clearly the map $\underline{f_G} : \underline{M_G} \longrightarrow \underline{G}$ is a right $\underline{\mathcal{X}}$-approximation of $\underline{G}$.

(ii) Clearly contravariant finiteness of $\mathcal{M}$ in $\mathsf{GProj}\,\mathscr{A}$ implies contravariant finiteness of $\underline{\mathcal{X}}$ in $\underline{\mathsf{GProj}}\,\mathscr{A}$. Conversely let $\underline{\mathcal{X}}$ be contravariantly finite in $\underline{\mathsf{GProj}}\,\mathscr{A}$ and let $G \in \mathsf{GProj}\,\mathscr{A}$. Let $\underline{f^G} : \underline{G} \longrightarrow \underline{X^G}$ be a left $\underline{\mathcal{X}}$-approximation of $\underline{G}$. Consider a map $f_1^G : G \longrightarrow M_1$, where $\underline{f_1^G} = \underline{f^G}$ and $\underline{M_1} = \underline{M^G}$. Since G is Gorenstein-projective, there exists a short exact sequence $0 \longrightarrow G \longrightarrow P^G \longrightarrow G' \longrightarrow 0$, where G' is Gorenstein-projective. Then the map $\mu : G \longrightarrow P^G$ is clearly a left projective approximation of G. It is easy to see that the map $f^G := (f_1^G, \mu) : G \longrightarrow M_1 \oplus P^G$ is a left $\mathcal{M}$-approximation of G.

(iii) By Remark 8.2 we have isomorphisms, $\forall G, G_1, G_2 \in \mathsf{GProj}\,\mathscr{A}$, $\forall A \in \mathscr{A}$, $\forall k \geqslant 1$:

$$\mathsf{Ext}^k(G, A) \xrightarrow{\cong} \underline{\mathsf{Hom}}(\Omega^k G, A) \qquad \text{and} \qquad \mathsf{Ext}^k(G_1, G_2) \xrightarrow{\cong} \underline{\mathsf{Hom}}(G_1, \Omega^{-k} G_2)$$

It follows directly that: $\underline{\mathcal{X}}_n^{\top} = \underline{\mathcal{X}}$ if and only if $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$.

(iv) Part (a) follows from (iii) since if $\mathscr{A}$ is Gorenstein, then $\mathsf{GProj}\,\mathscr{A}$ is contravariantly finite in $\mathscr{A}$ and (b) follows from (iii) since if $\mathsf{Proj}\,\mathscr{A}$ is covariantly finite in $\mathscr{A}$ and $\mathscr{A}$ is Gorenstein, then $\mathsf{GProj}\,\mathscr{A}$ is covariantly finite in $\mathscr{A}$, see [7]. $\square$

Let $\mathscr{A}$ be an abelian category with enough projectives, resp. injectives. Then $\mathscr{A}$ is called *(projectively)*, resp. *(injectively) Gorenstein* if there exists $n \geqslant 0$, such that any object of $\mathscr{A}$ admits a exact resolution of length $\leqslant n$ consisting of Gorenstein-projective, resp. Gorenstein-injective, objects. The minimum such n is called the *Gorenstein dimension* of $\mathscr{A}$, denoted by $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}$, and coincides with the Gorenstein dimension as defined in Section 7, if $\mathscr{A}$ has enough projectives and enough injectives, see [12].

Let $\mathcal{U}$, $\mathcal{V}$ be full subcategories of $\mathscr{A}$. Then we define $\mathcal{U} \diamond \mathcal{V}$ to be the full subcategory

$$\mathcal{U} \diamond \mathcal{V} = \mathsf{add}\{A \in \mathscr{A} \mid \exists \text{ an exact sequence}: \ 0 \longrightarrow U \longrightarrow A \longrightarrow V \longrightarrow 0, \ \text{ where } U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$$

Inductively we define $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$, $\forall n \geqslant 1$, for full subcategories $\mathcal{U}_i$ of $\mathscr{A}$. Clearly the operation $\diamond$ is associative and clearly $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$ coincides with the full subcategory $\mathsf{Filt}(\mathcal{U}_1, \cdots, \mathcal{U}_n)$ of $\mathscr{A}$ consisting of direct summands of objects A which admit a finite filtration

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$$

such that $A_k/A_{k-1} \in \mathcal{U}_k$, $1 \leqslant k \leqslant n$. Hence: $\mathsf{Filt}(\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n) = \mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$.

If $\mathcal{M}$ is contained in $\mathsf{GProj}\,\mathscr{A}$, then we denote by $\Omega^{-1}\mathcal{M}$ the full subcategory of $\mathsf{GProj}\,\mathscr{A}$ consisting of all direct summands of objects A for which there exists an exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$, where $M \in \mathcal{M}$ and P is projective. Then $\Omega^{-k}\mathcal{M}$ is defined inductively for $k \geqslant 2$.

Now we are ready to prove the main result of this section.

Theorem 10.6. *Let $\mathscr{A}$ be an abelian category with enough projectives. Let $\underline{\mathcal{X}}$ be a full subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$ and set $\mathcal{M} = \pi^{-1}\underline{\mathcal{X}}$. Then the following are equivalent.*

(i) *$\mathscr{A}$ is Gorenstein and $\underline{\mathcal{X}}$ is an $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$.*

(ii) *$\mathcal{M}$ is contravariantly finite in $\mathscr{A}$ and $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$ and $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} < \infty$.*

If (i) *holds and $\underline{\mathcal{X}} \neq 0$, then ${}^{\perp}_{n}\mathcal{M} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$, we have an equality*

$$\mathscr{A} = \mathcal{M} \diamond \Omega^{-1}\mathcal{M} \diamond \cdots \diamond \Omega^{-n}\mathcal{M} \diamond \mathsf{Proj}^{\leqslant d}\mathscr{A} \tag{10.5}$$

where $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} = d$, and $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M}$ is bounded as follows:

$$n+2 \;\leqslant\; \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \;\leqslant\; \max\{n, \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}\} + 3 \tag{10.6}$$

Moreover $\mathsf{pd}_{\mathsf{mod}\text{-}\mathcal{M}} F = n+2$, $\forall F \in \mathsf{mod}\text{-}\underline{\mathcal{X}}$, $F \neq 0$, and:

(a) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} < n$, then: $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = n+2$.*

(b) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} = n$, then: $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \in \{n+2, n+3\}$.*

(c) *If $\mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} > n$, then: $n+2 \leqslant \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A} + 3$.*

Proof. (ii) $\Rightarrow$ (i) Contravariant finiteness of $\mathcal{M}$ in $\mathscr{A}$ implies that $\mathcal{M}$ has weak kernels, so $\mathsf{mod}\text{-}\mathcal{M}$ is abelian. Assume that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = t < \infty$. Then $\mathsf{pd}\,\mathbb{H}(A) \leqslant t$ and since $\mathcal{M}$ contains the projectives, this implies that any object A in $\mathscr{A}$ admits a finite $\mathcal{M}$-resolution of length $\leqslant t$. Since $\mathcal{M}$ consists of Gorenstein-projectives, it follows that the Gorenstein dimension of $\mathscr{A}$ is at most t, hence $\mathscr{A}$ is Gorenstein by [12]. Since $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$, by Lemma 10.5 we have $\underline{\mathcal{X}}_n^{\top} = \underline{\mathcal{X}}$, so $\underline{\mathcal{X}}$ is an $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$.

(i) $\Rightarrow$ (ii) Since $\underline{\mathcal{X}}$ is $(n+1)$-cluster tilting and $\mathscr{A}$ is Gorenstein, it follows from Lemma 10.5 that $\mathcal{M}$ is contravariantly finite in $\mathscr{A}$ and therefore $\mathcal{M}$ has weak kernels. Then $\mathsf{mod}\text{-}\mathcal{M}$ is abelian. We use throughout the restricted Yoneda functor

$$\mathbb{H} \,:\, \mathscr{A} \longrightarrow \mathsf{mod}\text{-}\mathcal{M}, \quad \mathbb{H}(A) = \mathscr{A}(-,A)|_{\mathcal{M}}$$

Since by Lemma 10.5 we have $\mathcal{M}_n^{\perp} \cap \mathsf{GProj}\,\mathscr{A} = \mathcal{M}$, it suffices to show that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} < \infty$. We show first that $\mathsf{pd}\,\mathbb{H}(G) \leqslant n$, $\forall G \in \mathsf{GProj}\,\mathscr{A}$. Since $\underline{\mathcal{X}} = \underline{\mathcal{M}}$ is an $(n+1)$-cluster tilting subcategory of $\mathsf{GProj}\,\mathscr{A}$, it follows by Theorem 6.3 that $\underline{\mathsf{GProj}}\,\mathscr{A} = \underline{\mathcal{X}} \star \Omega^{-1}\underline{\mathcal{X}} \star \cdots \star \Omega^{-n}\underline{\mathcal{X}}$. We show by induction on n that for any Gorenstein-projective object G there exists an exact resolution

$$0 \longrightarrow M^n \longrightarrow M^{n-1} \longrightarrow \cdots \longrightarrow M^1 \longrightarrow M^0 \longrightarrow G \longrightarrow 0 \tag{10.7}$$

of G by objects from $\mathcal{M}$. If $n = 1$, then $\underline{\mathsf{GProj}}\,\mathscr{A} = \underline{\mathcal{X}} \star \Omega^{-1}\underline{\mathcal{X}}$, hence there exists a triangle $\underline{X}^1 \longrightarrow \underline{X}^0 \longrightarrow \underline{G} \longrightarrow \Omega^{-1}\underline{X}^1$. By the construction of triangles in $\underline{\mathsf{GProj}}\,\mathscr{A}$, the above triangle is induced by a short exact sequence $0 \longrightarrow M^1 \longrightarrow M^0 \longrightarrow G \longrightarrow 0$, where $\underline{M}^1 = \underline{X}^1$ and $\underline{M}^0 = \underline{X}^0$, so the M^i lie in $\mathcal{M}$. If $n = 2$, then $\underline{\mathsf{GProj}}\,\mathscr{A} = \underline{\mathcal{X}} \star \Omega^{-1}\underline{\mathcal{X}} \star \Omega^{-2}\underline{\mathcal{X}}$, and therefore there exists a triangle $\underline{X}^0 \longrightarrow \underline{G} \longrightarrow \underline{G}^1 \longrightarrow \Omega^{-1}\underline{X}^0$, where $\underline{X}^0$ lies in $\underline{\mathcal{X}}$ and $\underline{G}^1$ lies in $\Omega^{-1}\underline{\mathcal{X}} \star \Omega^{-2}\underline{\mathcal{X}}$. Then as before, there exists a short exact sequence $0 \longrightarrow A^1 \longrightarrow M^0 \longrightarrow G \longrightarrow 0$, where $\underline{A}^1 = \Omega\underline{G}^1$ and $\underline{M}^0 = \underline{X}^0$. Since $\Omega\underline{G}^1$ lies in $\underline{\mathcal{X}} \star \Omega^{-1}\underline{\mathcal{X}}$, it follows that there exists a short exact sequence $0 \longrightarrow M^2 \longrightarrow M^1 \longrightarrow A^1 \longrightarrow 0$, where $\underline{X}^i = \underline{M}^i$. We infer that there exists a short exact sequence $0 \longrightarrow M^2 \longrightarrow M^1 \longrightarrow M^0 \longrightarrow G \longrightarrow 0$, where the M^i lie in $\mathcal{M}$. Continuing by induction we have the short exact sequence (10.7). Applying $\mathbb{H}$ to the exact resolution (10.7) and using that $\mathcal{M}$ is n-rigid, we infer that

$$0 \longrightarrow \mathbb{H}(M^n) \longrightarrow \mathbb{H}(M^{n-1}) \longrightarrow \cdots \longrightarrow \mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow \mathbb{H}(G) \longrightarrow 0 \tag{10.8}$$

is exact, so it is a projective resolution of $\mathbb{H}(G)$ in $\mathsf{mod}\text{-}\mathcal{M}$. Hence $\mathsf{pd}\,\mathbb{H}(G) \leqslant n$.

Next we show that $\mathsf{pd}\,\mathbb{H}(A) \leqslant \max\{n, \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}\} + 1$, $\forall A \in \mathscr{A}$. Indeed let $t := \mathsf{G}\text{-}\mathsf{dim}\,\mathscr{A}$. Consider the cotorsion pair $(\mathsf{GProj}\,\mathscr{A}, \mathsf{Proj}^{<\infty}\mathscr{A})$ in $\mathscr{A}$ and note that $\mathsf{Proj}^{<\infty}\mathscr{A} = \mathsf{Proj}^{\leqslant t}\mathscr{A}$, see [12]. Let $0 \longrightarrow Y_A \longrightarrow G_A \longrightarrow A \longrightarrow 0$ be the right $\mathsf{GProj}\,\mathscr{A}$-approximation sequence of A, so Y_A has finite projective dimension, i.e. $\mathsf{pd}\,Y_A \leqslant t$. Applying the functor $\mathsf{Hom}_{\mathscr{A}}(\mathcal{M},-)$ to the above short exact sequence, and using that $\mathsf{Ext}^n_{\mathscr{A}}(\mathcal{M}, Y_A) = 0$, $\forall n \geqslant 1$, since $\mathcal{M}$ consists of Gorenstein-projective objects and Y_A has finite projective dimension, we deduce a short exact sequence

$$0 \longrightarrow \mathbb{H}(Y_A) \longrightarrow \mathbb{H}(G_A) \longrightarrow \mathbb{H}(A) \longrightarrow 0 \tag{10.9}$$

in $\mathsf{mod}\text{-}\mathcal{M}$. Let $0 \longrightarrow P^t \longrightarrow P^{t-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow Y_A \longrightarrow 0$ be a projective resolution of Y_A. Applying the functor $\mathbb{H}$, and using that $\mathcal{M}$ consists of Gorenstein-projective objects, we then have an exact sequence

$$0 \longrightarrow \mathbb{H}(P^t) \longrightarrow \mathbb{H}(P^{t-1}) \longrightarrow \cdots \longrightarrow \mathbb{H}(P^1) \longrightarrow \mathbb{H}(P^0) \longrightarrow \mathbb{H}(Y_A) \longrightarrow 0 \tag{10.10}$$

which is a projective resolution of $\mathbb{H}(Y_A)$. Hence $\mathsf{pd}\,\mathbb{H}(Y_A) \leqslant t$. Now since $\mathsf{pd}\,\mathbb{H}(G_A) \leqslant n$ and $\mathsf{pd}\,\mathbb{H}(Y_A) \leqslant t$, it follows from (10.9) that $\mathsf{pd}\,\mathbb{H}(A) \leqslant \max\{n, t\} + 1$. We infer that $\mathsf{pd}\,\mathbb{H}(A) \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 1$, $\forall A \in \mathscr{A}$. Next we show that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3$. For any object $F \in \mathsf{mod}\text{-}\mathcal{M}$, considering the exact sequence (10.2) associated to F, it follows directly that $\mathsf{pd}\,F \leqslant \mathsf{pd}\,\mathbb{H}(A_F) + 2$. Hence by the above we have $\mathsf{pd}\,F \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3$. Thus $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3 < \infty$.

Now assume that (i) holds and $\underline{\mathcal{X}} \neq 0$, or equivalently $\mathcal{M} \neq \mathsf{Proj}\,\mathscr{A}$. Then Proposition 10.3 shows that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \geqslant n + 2$. Hence we have the bounds $n + 2 \leqslant \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{X} \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3$. If $0 \neq F \in \mathsf{mod}\text{-}\mathcal{X}$, then F admits a presentation $0 \longrightarrow \mathbb{H}(A_F) \longrightarrow \mathbb{H}(M^1) \longrightarrow \mathbb{H}(M^0) \longrightarrow F \longrightarrow 0$, where the map $M^1 \longrightarrow M^0$ is epic. Since the M^i are Gorenstein-projective and $\mathsf{GProj}\,\mathscr{A}$ is closed under kernels of epimorphisms, it follows that A_F is Gorenstein-projective. Hence by the above argument we have $\mathsf{pd}\,\mathbb{H}(A_F) \leqslant n$ and therefore $\mathsf{pd}\,F \leqslant n+2$. Since by Lemma 10.2(iii), $\mathsf{Ext}^k(F, \mathbb{H}(\mathcal{M})) = 0$, $0 \leqslant k \leqslant n+1$, and $F \neq 0$, we infer by Lemma 10.1 that $\mathsf{pd}\,F = n+2$.

We show equation (10.5). Let A be in $\mathscr{A}$ and consider the exact sequence $0 \longrightarrow Y_A \longrightarrow G_A \longrightarrow A \longrightarrow 0$ where the map $G_A \longrightarrow A$ is a right $\mathsf{GProj}\,\mathscr{A}$-approximation sequence of A, so Y_A has finite projective dimension, i.e. $Y_A \in \mathsf{Proj}^{<\infty}\,\mathscr{A}$. Since G_A is Gorenstein-projective, there exists a short exact sequence $0 \longrightarrow G_A \longrightarrow P \longrightarrow \Omega^{-1}G_A \longrightarrow 0$, where P is projective and $\Omega^{-1}G_A$ is Gorenstein-projective. Then the composition $Y_A \longrightarrow G_A \longrightarrow P$ induces a short exact sequence $0 \longrightarrow Y_A \longrightarrow P \longrightarrow Y^A \longrightarrow 0$ and clearly Y^A has finite projective dimension. By diagram chasing then it is easy to see that there exists a short exact sequence $0 \longrightarrow G_A \longrightarrow A \oplus P \longrightarrow Y^A \longrightarrow 0$. Hence $A \in \mathsf{GProj}\,\mathscr{A} \diamond \mathsf{Proj}^{<\infty}\,\mathscr{A}$. Using that $\underline{\mathsf{GProj}}\,\mathscr{A} = \mathcal{X} \star \mathcal{X}[1] \star \cdots \star \mathcal{X}[n]$, it follows that $\mathsf{GProj}\,\mathscr{A} = \mathcal{M} \diamond \Omega^{-1}\mathcal{M} \diamond \cdots \diamond \Omega^{-n}\mathcal{M}$. We infer that $\mathscr{A} = \mathsf{GProj}\,\mathscr{A} \diamond \mathsf{Proj}^{<\infty}\,\mathscr{A} = \mathcal{M} \diamond \Omega^{-1}\mathcal{M} \diamond \cdots \diamond \Omega^{-n}\mathcal{M} \diamond \mathsf{Proj}^{<\infty}\,\mathscr{A}$.

(a) If $\mathsf{G\text{-}dim}\,\mathscr{A} < n$, then any object $Y \in \mathscr{A}$ of finite projective dimension has projective dimension $< n$. It follows that $\mathsf{pd}\,\mathbb{H}(Y) < n$. Now the short exact sequence (10.9) induces a long exact sequence, $\forall A \in \mathscr{A}$:

$$\cdots \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(\mathbb{H}(G_A), -) \longrightarrow \mathsf{Ext}^n_{\mathscr{A}}(\mathbb{H}(Y_A), -) \longrightarrow \mathsf{Ext}^{n+1}_{\mathscr{A}}(\mathbb{H}(A), -) \longrightarrow \mathsf{Ext}^{n+1}_{\mathscr{A}}(\mathbb{H}(G_A), -) \longrightarrow \cdots$$

Since we have $\mathsf{pd}\,\mathbb{H}(G_A) \leqslant n$ and $\mathsf{pd}\,\mathbb{H}(Y_A) < n$, we have $\mathsf{Ext}^{n+1}_{\mathscr{A}}(\mathbb{H}(A), -) = 0$ and therefore $\mathsf{pd}\,\mathbb{H}(A) \leqslant n$. It follows that $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant n+2$ and consequently $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = n+2$.

(b) If $\mathsf{G\text{-}dim}\,\mathscr{A} = n$, then as above we have $\mathsf{Ext}^{n+2}_{\mathscr{A}}(\mathbb{H}(A), -) = 0$, so $\mathsf{pd}\,\mathbb{H}(A) \leqslant n+1$ and therefore $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant n+3$. Then from (10.6) we have $n+2 \leqslant \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant n+3$.

(c) If $\mathsf{G\text{-}dim}\,\mathscr{A} > n$, then as above we have $\mathsf{Ext}^{\mathsf{G\text{-}dim}\,\mathscr{A}+2}_{\mathscr{A}}(\mathbb{H}(A), -) = 0$, so $\mathsf{pd}\,\mathbb{H}(A) \leqslant \mathsf{G\text{-}dim}\,\mathscr{A} + 1$ and therefore $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant \mathsf{G\text{-}dim}\,\mathscr{A} + 3$. Then from (10.6) we have $n+2 \leqslant \mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} \leqslant \mathsf{G\text{-}dim}\,\mathscr{A} + 3$. □

Corollary 10.7. *Let $\mathscr{A}$ be a Gorenstein abelian category with enough projectives. Let $\underline{\mathcal{X}}$ be an $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$ and let $\mathcal{M} = \pi^{-1}\underline{\mathcal{X}}$. Then for any $F \in \mathsf{mod}\text{-}\mathcal{X}$ and any $G \in \mathsf{mod}\text{-}\mathcal{X}$ we have:*

$$\mathsf{Ext}^k(F, G) = 0, \quad \forall k \geqslant 0, \quad k \neq n+2-i, \cdots, n+2, \quad 0 \leqslant i = \mathsf{pd}\,G \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3$$

Proof. By Theorem 10.6 we have $\mathsf{pd}\,F = n+2$ and $\mathsf{Ext}^k(F, \mathbb{H}(\mathcal{M})) = 0$, $\forall k \geqslant 0$, $k \neq n+2$. On the other by the same Theorem we know that $\mathsf{pd}\,G \leqslant \max\{n, \mathsf{G\text{-}dim}\,\mathscr{A}\} + 3$. Now the assertion follows easily by induction on $\mathsf{pd}\,G$ by applying the functor $(F, -)$ to a the extensions $0 \longrightarrow \Omega^j G \longrightarrow \mathbb{H}(N^{j-1}) \longrightarrow \Omega^{j-1} G \longrightarrow 0$. □

Corollary 10.8. *Let $\mathscr{A}$ be Gorenstein and assume that $\underline{\mathsf{GProj}}\,\mathscr{A}$ is $(n+1)$-Calabi-Yau. Let $\underline{\mathcal{X}}$ be an $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$ and let $\mathcal{M} = \pi^{-1}\underline{\mathcal{X}}$. If $\underline{\mathsf{Hom}}(\mathcal{M}, \Omega^i\mathcal{M}) = 0$, $1 \leqslant i \leqslant n-1$, then $\mathsf{mod}\text{-}\mathcal{X}$ is 1-Gorenstein and the stable triangulated category $\underline{\mathsf{GProj}}\,\mathsf{mod}\text{-}\mathcal{X}$ is $(n+2)$-Calabi-Yau.*

Keller and Reiten proved that if $\mathcal{X}$ is a 2-cluster tilting subcategory of $\underline{\mathscr{A}}$, where $\mathscr{A}$ is a Frobenius abelian category and if $\underline{\mathscr{A}}$ is 2-Calabi-Yau, then $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = 3$, where $\mathcal{M} = \pi^{-1}(\underline{\mathcal{X}})$, see [21]. The following direct consequence of Theorem 10.6 and Corollary 10.4, generalizes the result of Keller-Reiten for any $(n+1)$-cluster tilting subcategory, $n \geqslant 2$, without assuming the Calabi-Yau condition.

Corollary 10.9. *Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. For a full subcategory $\mathcal{M} \subseteq \mathsf{GProj}\,\mathscr{A}$, the following are equivalent.*

(i) *$\mathscr{A}$ is Frobenius and $\underline{\mathcal{M}}$ is an $(n+1)$-cluster tilting subcategory of $\underline{\mathscr{A}}$.*
(ii) *$\mathcal{M}$ is contravariantly finite in $\mathscr{A}$, contains the projectives, and $\mathcal{M}^{\perp}_{n} = \mathcal{M}$.*
(iii) *$\mathcal{M}$ is covariantly finite in $\mathscr{A}$, contains the injectives, and ${}^{\perp}_{n}\mathcal{M} = \mathcal{M}$.*
(iv) *$\mathcal{M}$ is n-rigid, contravariantly in $\mathscr{A}$ and contains the projectives, or covariantly finite and contains the injectives, and* $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} = n+2$.

In particular if $\mathscr{A}$ is a Krull-Schmidt Frobenius abelian category and $\mathcal{X}$ is an $(n+1)$-cluster subcategory of $\underline{\mathscr{A}}$ which is of finite representation type, then

$$\mathsf{rep.dim}\,\mathscr{A} \leqslant n+2$$

Corollary 10.10. *For an Artin algebra Λ the following are equivalent.*

(i) *Λ is Gorenstein and $\underline{\mathsf{Gproj}}\,\Lambda$ contains a $(n+1)$-cluster tilting object.*

(ii) *$\mathscr{A}$ contains a generator M such that $\mathsf{add}\,M = M_n^{\perp} \cap \mathsf{Gproj}\,\Lambda$ and $\mathsf{gl.dim}\,\mathsf{End}_\Lambda(M) < \infty$.*

If (ii) holds, then M is a cogenerator if and only if Λ is self-injective and then $\mathsf{rep.dim}\,\Lambda \leqslant n+2$.

Corollary 10.11. *Let Λ be an Artin algebra and let $\underline{T}$ be an $(n+1)$-cluster tilting object of $\underline{\mathsf{Gproj}}\,\Lambda$, $n \geqslant 1$. If $\mathsf{Ext}^t(T, \mathsf{DTr}T) = 0$, $2 \leqslant t \leqslant n-k+1$, where $0 \leqslant k \leqslant n-1$, then the cluster tilting algebra $\mathsf{mod}\text{-}\underline{\mathsf{End}}_\Lambda(T)$ is k-Gorenstein. Moreover for $0 \leqslant k \leqslant 1$, the stable category $\underline{\mathsf{Gproj}}\,\mathsf{mod}\text{-}\underline{\mathsf{End}}_\Lambda(T)$ is $(n+2)$-Calabi-Yau provided that $\underline{\mathsf{Gproj}}\,\Lambda$ is $(n+1)$-Calabi-Yau.*

Proof. For any $T_1, T_2 \in \mathsf{add}\,T$ and for $2 \leqslant t \leqslant n-k+1$, we have:

$$\mathsf{Ext}^t_\Lambda(T_1, \mathsf{DTr}T_2) \;\cong\; \mathsf{Ext}^1_\Lambda(\Omega^{t-1}T_1, \mathsf{DTr}T_2) \;\cong\; \mathsf{D}\underline{\mathsf{Hom}}_\Lambda(T_2, \Omega^{t-1}T_1)$$

It follows that $\underline{\mathsf{Hom}}_\Lambda(T_2, \Omega^t T_1) = 0$, $1 \leqslant t \leqslant n-k$ and therefore the $(n+1)$-cluster tilting object $\underline{T}$ is $(n-k)$-strong. Then the assertions follow from Corollary 10.8. □

The following is also a result of Keller-Reiten, see [21, Theorem 5.4], called *relative $(n+2)$-Calabi-Yau duality*, proved in loc.cit., in the setting of an algebraic $(n+1)$-Calabi-Yau triangulated category over a field. However in our setting we give a different proof.

Corollary 10.12. *Let $\mathscr{A}$ be a Gorenstein abelian Hom-finite k-category over a field k, and assume that $\underline{\mathsf{GProj}}\,\mathscr{A}$ is $(n+1)$-Calabi-Yau. Let $0 \neq \mathcal{X}$ be a $(n+1)$-cluster tilting subcategory of $\underline{\mathsf{GProj}}\,\mathscr{A}$, where $d \geqslant 2$. If $\mathcal{M} = \pi^{-1}\mathcal{X}$, then for any object $F \in \mathsf{mod}\text{-}\mathcal{X} \subseteq \mathsf{mod}\text{-}\mathcal{M}$, there is a natural isomorphism:*

$$\mathsf{D}\,\mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(F, -\big) \xrightarrow{\cong} \mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(-, F[n+2]\big)$$

In particular, for any two objects $F, G \in \mathsf{mod}\text{-}\mathcal{M}$ with $F \in \mathsf{mod}\text{-}\mathcal{X} \subseteq \mathsf{mod}\text{-}\mathcal{M}$:

$$\mathsf{DExt}^i_{\mathsf{mod}\text{-}\mathcal{M}}\big(F, G\big) \xrightarrow{\cong} \mathsf{Ext}^{n-i+2}_{\mathsf{mod}\text{-}\mathcal{M}}\big(G, F\big), \quad i \in \mathbb{Z}$$

Proof. By Theorem 1.5, $\mathsf{gl.dim}\,\mathsf{mod}\text{-}\mathcal{M} < \infty$, so by a result of Happel [14], $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})$, which coincides, up to equivalence, with the category of perfect complexes over $\mathsf{mod}\text{-}\mathcal{M}$, has Auslander-Reiten triangles and therefore $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})$ admits a Serre functor which is given by $\mathcal{S} = - \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}[-1]$, where $\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}(M) = \mathsf{Hom}_{\mathscr{A}}(-,M) = \mathbb{H}(M)$. Hence we have a natural isomorphism

$$\mathsf{D}\,\mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(F, -\big) \xrightarrow{\cong} \mathsf{Hom}_{\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})}\big(-, F \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}[-1]\big)$$

So it suffices to show that we have an isomorphism $F[n+2] \xrightarrow{\cong} F \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}[-1]$, or equivalently an isomorphism in the derived category $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})$:

$$F[n+3] \xrightarrow{\cong} F \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}} \tag{10.11}$$

Applying the triangulated functor $- \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}$ to projective resolution of F in (10.4) with F deleted, it is easy to see that we obtain a complex

$$0 \longrightarrow \mathsf{DHom}_{\mathscr{A}}(M^n_{A_F}, -)|_{\mathcal{M}} \longrightarrow \mathsf{DHom}_{\mathscr{A}}(M^{n-1}_{A_F}, -)|_{\mathcal{M}} \longrightarrow \cdots \longrightarrow \mathsf{DHom}_{\mathscr{A}}(M^0_{A_F}, -)|_{\mathcal{M}} \longrightarrow$$

$$\cdots \longrightarrow \mathsf{DHom}_{\mathscr{A}}(M^1, -)|_{\mathcal{M}} \longrightarrow \mathsf{DHom}_{\mathscr{A}}(M^0, -)|_{\mathcal{M}} \longrightarrow 0$$

which is acyclic everywhere except in first position on the left, which corresponds to $-n-3$ degree, where the homology is given by $\mathsf{DExt}^1(K_F^{n-1}, -)|_{\mathcal{M}}$. Since $\underline{\mathsf{GProj}}\,\mathscr{A}$ is $(n+1)$-Calabi-Yau, we have natural isomorphisms:

$$\mathsf{DExt}^1(K_F^{n-1}, -)|_{\mathcal{M}} \cong \mathsf{D}\underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega K_F^{n-1}, -)|_{\mathcal{M}} \cong \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-n-1}\Omega K_F^{n-1})|_{\mathcal{M}} =$$

$$\underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-n} K_F^{n-1})|_{\mathcal{M}} \cong \underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega^n(-), K_F^{n-1})|_{\mathcal{M}} \cong \mathsf{Ext}^n_{\mathscr{A}}(-, K_F^{n-1})|_{\mathcal{M}} \cong F$$

where the last isomorphism follows from Lemma 10.2. We infer that in $\mathbf{D}^b(\mathsf{mod}\text{-}\mathcal{M})$ we have isomorphisms

$$F[n+3] \xrightarrow{\cong} \mathsf{Ext}^n_{\mathscr{A}}(-, K_F^{n-1})|_{\mathcal{M}} \xrightarrow{\cong} F \otimes^{\mathbf{L}}_{\mathcal{M}} \mathsf{D}\,\mathsf{Hom}_{\mathscr{A}}(-,?)|_{\mathcal{M}}$$ □

References

[1] C. Amiot, *On the structure of triangulated categories with finitely many indecomposables*, Bull. Soc. Math. France **135** (2007), no. 3, 435–474.

[2] M. Auslander, *Coherent Functors*, in: Proceedings of the Conference on Categorical Algebra, La Jolla (1966), 189–231.

[3] ———, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes, London (1971).

[4] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., No. **94** (1969), 146 pp.

[5] ——— and I. Reiten, *Stable Equivalence of Dualizing R-Varieties*, Advances in Math. **12** (1974), 306–366.

[6] ———, ———, and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press, (1995).

[7] A. Beligiannis, *The Homological Theory of Contravariantly Finite Subcategories: Aus–lander-Buchweitz Contexts, Gorenstein Categories and (Co-)Stabilizations*, Comm. Algebra **28**(10), (2000), 4547–4596.

[8] ———, *Relative Homological Algebra and Purity in Triangulated Categories*, J. Algebra **227** (2000), 268–361.

[9] ———, *Cohen-Macaulay Modules, (Co)Torsion Pairs, and Virtually Gorenstein Algebras*, J. Algebra Vol. **288**, No.1, (2005), 137–211.

[10] ———, *On Algebras of Finite Cohen-Macaulay Tyep*, preprint, (2008).

[11] ——— and N. Marmaridis, *Left Triangulated Categories Arising from Contravariantly Finite Subcategories*, Comm. in Algebra **22** (1994), 5021–5036.

[12] ——— and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207 pp.

[13] A. Bondal and M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Math. USSR-Izv. **35** (1990), no. 3, 519–541.

[14] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge, 1988. x+208 pp.

[15] O. Iyama, *Higher Auslander-Reiten Theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), 22–50.

[16] ——— and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen- Macaulay modules*, Invent. Math. **172** (2008), 117–168.

[17] ——— and S. Oppermann, *Stable categories of higher preprojective algebras*, preprint (2009).

[18] P. Joergensen and Th. Holm, *On the relation between cluster and classical tilting*, J. Pure Appl. Algebra **214**, (2010), 1523-1533.

[19] B. Keller, *Derived categories and universal problems*, Comm. Algebra **19** (1991), no. 3, 699–747.

[20] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581.

[21] ——— and I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

[22] ——— and ———, *Acyclic Calabi-Yau Categories*, to appear in Compositio Math.

[23] S. Koenig and B. Zhu, *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. **258** (2008), no. 1, 143–160.

[24] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, (Russian. Russian summary) Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 240–262; translation in Proc. Steklov Inst. Math. 2004, no. 3 (246), 227–248.

[25] Y. Palu, *Grothendieck group and generalized mutation rule for 2-Calabi-Yau triangulated categories*, J. Pure Appl. Algebra **213** (2009), no. 7, 1438–1449.

[26] ———, *Dimensions of triangulated categories*, Journal of K-theory **1** (2008), 193-256 and errata, 257-258.

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